2 Dynamics

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Inside

1. Fundamentals

1.1 Space and Time

Space is the three-dimensional universe. The distance between two points in space is the length of the straight line joining them. The unit of length in the International System of units, or SI units, is the meter (m). In U.S. customary units, the unit of length is the foot (ft). The U.S. customary units use the mile (mi) (1 mi = 5280 ft) and the inch (in) (1 ft = 12 in).

The *time* is a scalar and is measured by the intervals between repeatable events. The unit of time is the second (s) in both SI units and U.S. customary units. The minute (min), hour (hr), and day are also used.

The *velocity* of a point in space relative to some reference is the rate of change of its position with time. The velocity is expressed in meters per second (m/s) in SI units, and is expressed in feet per second (ft/s) in U.S. customary units.

The *acceleration* of a point in space relative to some reference is the rate of change of its velocity with time. The acceleration is expressed in meters per second squared (m/s^2) in SI units, and is expressed in feet per second squared (ft/s^2) in U.S. customary units.

1.2 Numbers

Engineering measurements, calculations, and results are expressed in numbers. Significant digits are the number of meaningful digits in a number, counting to the right starting with the first nonzero digit. Numbers can be rounded off to a certain number of significant digits. The value of π can be expressed to three significant digits, 3.14, or can be expressed to six significant digits, 3.14159.

The multiples of units are indicated by prefixes. The common prefixes, their abbreviations, and the multiples they represent are shown in Table 1.1. For example, 5 km is 5 kilometers, which is 5000 m.

Table 1.1 *Prefixes Used in SI Units*

Prefix	Abbreviation	Multiple
nano-	n	10^{-9}
micro-	μ	10^{-6}
mili-	m	10^{-3}
kilo-	k	10^{3}
mega-	M	10^{6}
giga-	G	10^{9}

Table 1.2 *Unit Conversions*

Time	1 minute	=	60 seconds
	1 hour	=	60 minutes
	1 day	=	24 hours
Length	1 foot	=	12 inches
	1 mile	=	5280 feet
	1 inch	=	25.4 millimeters
	1 foot	=	0.3048 meter
Angle	2π radians	=	360 degrees

Some useful unit conversions are presented in Table 1.2. For example, 1 mi/hr in terms of ft/s is (1 mi equals 5280 ft and 1 hr equals 3600 s)

$$1\,\frac{\text{mi}}{\text{hr}} = 1\,\frac{1\text{ mi} = 5280\text{ ft}}{1\text{ hr} = 3600\text{ s}} = 1\,\frac{5280\text{ ft}}{3600\text{ s}} = 1.47\,\frac{\text{ft}}{\text{s}}.$$

1.3 Angular Units

Angles are expressed in radians (rad) in both SI and U.S. customary units. The value of an angle θ in radians (Fig. 1.1) is the ratio of the part of the

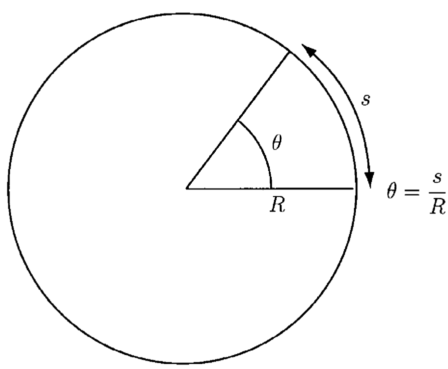

Figure 1.1

circumference s subtended by θ to the radius R of the circle,

$$\theta = \frac{s}{R}.$$

Angles are also expressed in degrees. There are 360 degrees (360°) in a complete circle. The complete circumference of the circle is $2\pi R$. Therefore,

$$360^\circ = 2\pi \text{ rad}.$$

2. Kinematics of a Point

2.1 Position, Velocity, and Acceleration of a Point

One may observe students and objects in a classroom, and their positions relative to the room. Some students may be in the front of the classroom, some in the middle of the classroom, and some in the back of the classroom. The classroom is the "frame of reference." One can introduce a cartesian coordinate system x, y, z with its axes aligned with the walls of the classroom. A reference frame is a coordinate system used for specifying the positions of points and objects.

The position of a point P relative to a given reference frame with origin O is given by the position vector $\mathbf{r}$ from point O to point P (Fig. 2.1). If the point P is in motion relative to the reference frame, the position vector $\mathbf{r}$ is a function of time t (Fig. 2.1) and can be expressed as

$$\mathbf{r} = \mathbf{r}(t).$$

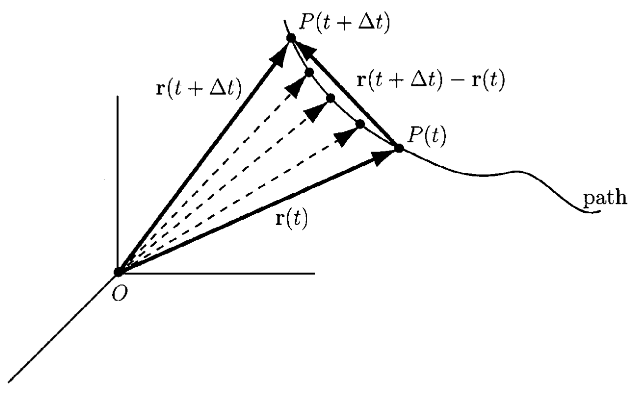

Figure 2.1

The velocity of the point P relative to the reference frame at time t is defined by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}, \tag{2.1}$$

where the vector $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ is the change in position, or displacement of P, during the interval of time Δt (Fig. 2.1). The velocity is the rate of change of the position of the point P. The magnitude of the velocity $\mathbf{v}$ is the speed $v = |\mathbf{v}|$. The dimensions of $\mathbf{v}$ are (distance)/(time). The position and velocity of a point can be specified only relative to a reference frame.

The acceleration of the point P relative to the given reference frame at time t is defined by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}, \tag{2.2}$$

where $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ is the change in the velocity of P during the interval of time Δt (Fig. 2.1). The acceleration is the rate of change of the velocity of P at time t (the second time derivative of the displacement), and its dimensions are (distance)/(time)^2.

2.2 Angular Motion of a Line

The angular motion of the line L, in a plane, relative to a reference line L_0, in the plane, is given by an angle θ (Fig. 2.2). The angular velocity of L relative to L_0 is defined by

$$\omega = \frac{d\theta}{dt} = \dot{\theta}, \tag{2.3}$$

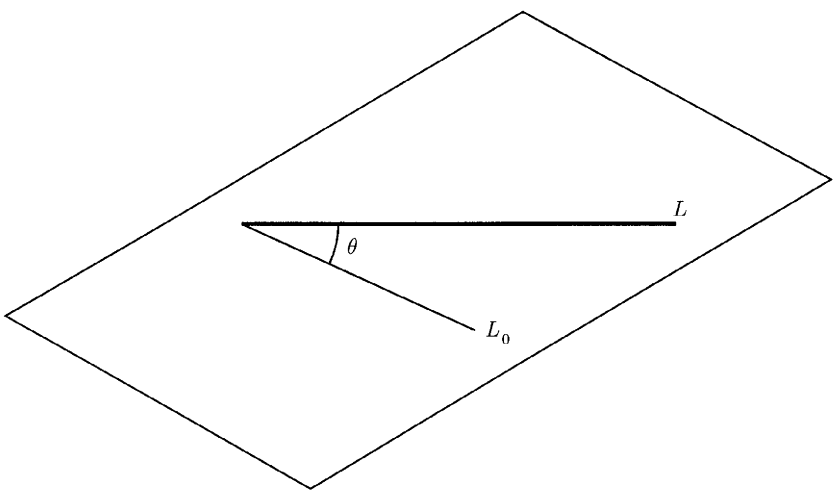

Figure 2.2

and the angular acceleration of L relative to L_0 is defined by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \dot{\omega} = \ddot{\theta}. \tag{2.4}$$

The dimensions of the angular position, angular velocity, and angular acceleration are (rad), (rad/s), and (rad/s^2), respectively. The scalar coordinate θ can be positive or negative. The counterclockwise (ccw) direction is considered positive.

2.3 Rotating Unit Vector

The angular motion of a unit vector **u** in a plane can be described as the angular motion of a line. The direction of **u** relative to a reference line L_0 is specified by the angle θ in Fig. 2.3a, and the rate of rotation of **u** relative to L_0 is defined by the angular velocity

$$\omega = \frac{d\theta}{dt}.$$

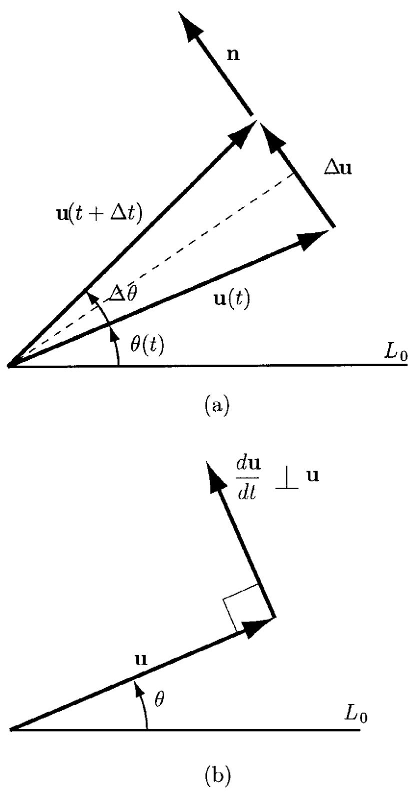

Figure 2.3

The time derivative of **u** is specified by

$$\frac{d\mathbf{u}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{u}(t+\Delta t) - \mathbf{u}(t)}{\Delta t}.$$

Figure 2.3a shows the vector **u** at time t and at time $t + \Delta t$. The change in **u** during this interval is $\Delta\mathbf{u} = \mathbf{u}(t+\Delta t)\mathbf{u}(t)$, and the angle through which **u** rotates is $\Delta\theta = \theta(t+\Delta t) - \theta(t)$. The triangle in Fig. 2.3a is isosceles, so the magnitude of $\Delta\mathbf{u}$ is

$$|\Delta\mathbf{u}| = 2|\mathbf{u}|\sin(\Delta\theta/2) = 2\sin(\Delta\theta/2).$$

The vector $\Delta\mathbf{u}$ is

$$\Delta\mathbf{u} = |\Delta\mathbf{u}|\mathbf{n} = 2\sin(\Delta\theta/2)\mathbf{n},$$

where $\mathbf{n}$ is a unit vector that points in the direction of $\Delta\mathbf{u}$ (Fig. 2.3a). The time derivative of $\mathbf{u}$ is

$$\begin{aligned}\frac{d\mathbf{u}}{dt} &= \lim_{\Delta t\to 0}\frac{\Delta\mathbf{u}}{\Delta t} = \lim_{\Delta t\to 0}\frac{2\sin(\Delta\theta/2)\mathbf{n}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\frac{\Delta\theta}{\Delta t}\mathbf{n}\\ &= \lim_{\Delta t\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\frac{\Delta\theta}{\Delta t}\mathbf{n} = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}\mathbf{n} = \frac{d\theta}{dt}\mathbf{n},\end{aligned}$$

where $\lim_{\Delta t\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2} = 1$ and $\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$. So the time derivative of the unit vector $\mathbf{u}$ is

$$\frac{d\mathbf{u}}{dt} = \frac{d\theta}{dt}\mathbf{n} = \omega\mathbf{n},$$

where $\mathbf{n}$ is a unit vector that is perpendicular to $\mathbf{u}$, $\mathbf{n} \perp \mathbf{u}$, and points in the positive θ direction (Fig. 2.3b).

2.4 Straight Line Motion

The position of a point P on a straight line relative to a reference point O can be indicated by the coordinate s measured along the line from O to P (Fig. 2.4). In this case the reference frame is the straight line and the origin of the reference frame is the point O. The reference frame and its origin are used to describe the position of point P. The coordinate s is considered to be positive to the right of the origin O and is considered to be negative to the left

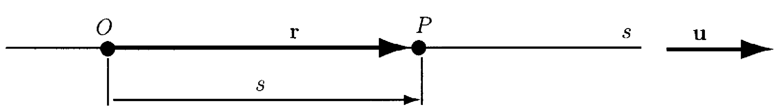

Figure 2.4

of the origin.

Let $\mathbf{u}$ be a unit vector parallel to the straight line and pointing in the positive s direction (Fig. 2.4). The position vector of the point P relative to the origin O is

$$\mathbf{r} = s\mathbf{u}.$$

The velocity of the point P relative to the origin O is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\mathbf{u} = \dot{s}\mathbf{u}.$$

The magnitude v of the velocity vector $\mathbf{v} = v\mathbf{u}$ is the speed (velocity scalar)

$$v = \frac{ds}{dt} = \dot{s}.$$

The speed v of the point P is equal to the slope at time t of the line tangent to the graph of s as a function of time.

The acceleration of the point P relative to O is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{u}) = \frac{dv}{dt}\mathbf{u} = \dot{v}\mathbf{u} = \ddot{s}\mathbf{u}.$$

The magnitude a of the acceleration vector $\mathbf{a} = a\mathbf{u}$ is the acceleration scalar

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The acceleration a is equal to the slope at time t of the line tangent to the graph of v as a function of time.

2.5 Curvilinear Motion

The motion of the point P along a curvilinear path, relative to a reference frame, can be specified in terms of its position, velocity, and acceleration vectors. The directions and magnitudes of the position, velocity, and acceleration vectors do not depend on the particular coordinate system used to express them. The representations of the position, velocity, and acceleration vectors are different in different coordinate systems.

2.5.1 CARTESIAN COORDINATES

Let $\mathbf{r}$ be the position vector of a point P relative to the origin O of a cartesian reference frame (Fig. 2.5). The components of $\mathbf{r}$ are the x, y, z coordinates of the point P,

$$\mathbf{r} = x\mathbf{\imath} + y\mathbf{\jmath} + z\mathbf{k}.$$

The velocity of the point P relative to the reference frame is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{dx}{dt}\mathbf{\imath} + \frac{dy}{dt}\mathbf{\jmath} + \frac{dz}{dt}\mathbf{k} = \dot{x}\mathbf{\imath} + \dot{y}\mathbf{\jmath} + \dot{z}\mathbf{k}. \tag{2.5}$$

The velocity in terms of scalar components is

$$\mathbf{v} = v_x\mathbf{\imath} + v_y\mathbf{\jmath} + v_z\mathbf{k}. \tag{2.6}$$

Three scalar equations can be obtained:

$$v_x = \frac{dx}{dt} = \dot{x}, \quad v_y = \frac{dy}{dt} = \dot{y}, \quad v_z = \frac{dz}{dt} = \dot{z}. \tag{2.7}$$

The acceleration of the point P relative to the reference frame is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{dv_x}{dt}\mathbf{\imath} + \frac{dv_y}{dt}\mathbf{\jmath} + \frac{dv_z}{dt}\mathbf{k} = \dot{v}_x\mathbf{\imath} + \dot{v}_y\mathbf{\jmath} + \dot{v}_z\mathbf{k} = \ddot{x}\mathbf{\imath} + \ddot{y}\mathbf{\jmath} + \ddot{z}\mathbf{k}.$$

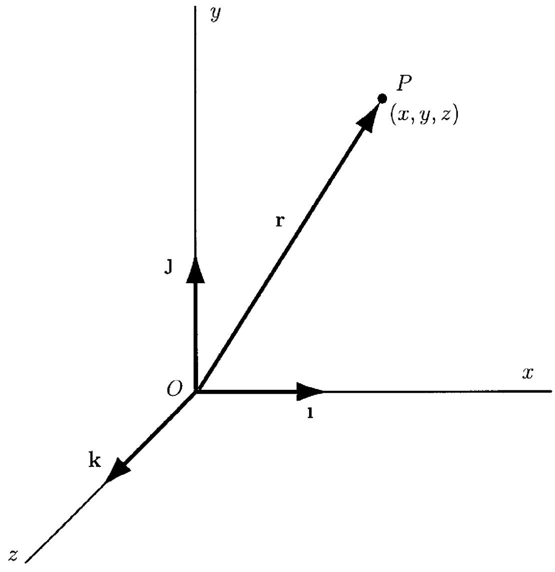

Figure 2.5

If we express the acceleration in terms of scalar components,

$$\mathbf{a} = a_x \mathbf{\imath} + a_y \mathbf{\jmath} + a_z \mathbf{k}, \tag{2.8}$$

three scalar equations can be obtained:

$$a_x = \frac{dv_x}{dt} = \dot{v}_x = \ddot{x}, \quad a_y = \frac{dv_y}{dt} = \dot{v}_y = \ddot{y}, \quad a_z = \frac{dv_z}{dt} = \dot{v}_z = \ddot{z}. \tag{2.9}$$

Equations (2.7) and (2.9) describe the motion of a point relative to a cartesian coordinate system.

2.6 Normal and Tangential Components

The position, velocity, and acceleration of a point will be specified in terms of their components tangential and normal (perpendicular) to the path.

2.6.1 PLANAR MOTION

The point P is moving along a plane curvilinear path relative to a reference frame (Fig. 2.6). The position vector $\mathbf{r}$ specifies the position of the point P relative to the reference point O. The coordinate s measures the position of the point P along the path relative to a point O' on the path. The velocity of P relative to O is

$$\Delta\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}, \tag{2.10}$$

where $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ (Fig. 2.6). The distance travelled along the path from t to $t + \Delta t$ is Δs. One can write Eq. (2.10) as

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \mathbf{u},$$

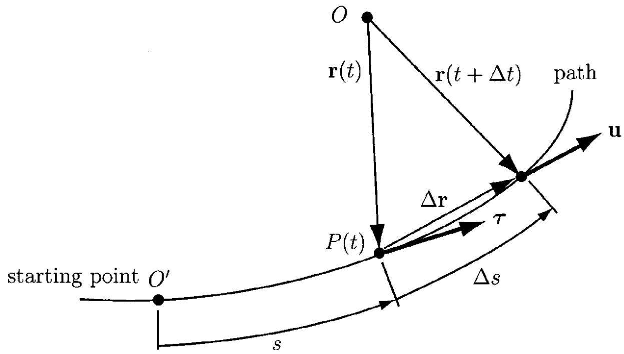

Figure 2.6

where **u** is a unit vector in the direction of $\Delta\mathbf{r}$. In the limit as Δt approaches zero, the magnitude of $\Delta\mathbf{r}$ equals ds because a chord progressively approaches the curve. For the same reason, the direction of $\Delta\mathbf{r}$ approaches tangency to the curve, and **u** becomes a unit vector, $\boldsymbol{\tau}$, tangent to the path at the position of P (Fig. 2.6):

$$\mathbf{v} = v\boldsymbol{\tau} = \frac{ds}{dt}\boldsymbol{\tau}. \tag{2.11}$$

The *tangent direction* is defined by the unit tangent vector $\boldsymbol{\tau}$, which is a path variable parameter

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\boldsymbol{\tau}$$

or

$$\boldsymbol{\tau} = \frac{d\mathbf{r}}{ds}. \tag{2.12}$$

The velocity of a point in curvilinear motion is a vector whose magnitude equals the rate of change of distance traveled along the path and whose direction is tangent to the path.

To determine the acceleration of P, the time derivative of Eq. (2.11) is taken:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\boldsymbol{\tau} + v\frac{d\boldsymbol{\tau}}{dt}. \tag{2.13}$$

If the path is not a straight line, the unit vector $\boldsymbol{\tau}$ rotates as P moves on the path, and the time derivative of $\boldsymbol{\tau}$ is not zero. The path angle θ defines the direction of $\boldsymbol{\tau}$ relative to a reference line shown in Fig. 2.7. The time derivative of the rotating tangent unit vector $\boldsymbol{\tau}$ is

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{d\theta}{dt}\boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is a unit vector that is normal to $\boldsymbol{\tau}$ and points in the positive θ direction if $d\theta/dt$ is positive. The normal unit vector $\boldsymbol{\nu}$ defines the normal

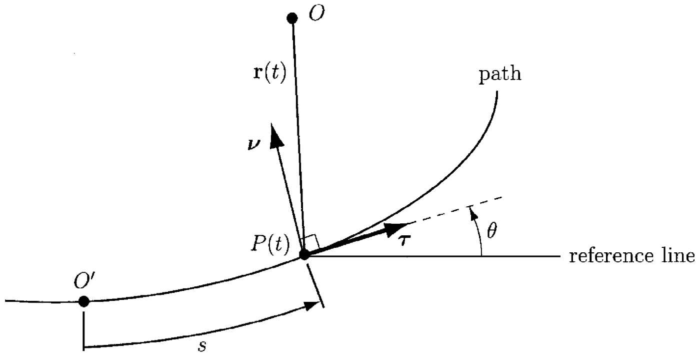

Figure 2.7

direction to the path. If we substitute this expression into Eq. (2.13), the acceleration of P is obtained:

$$\mathbf{a} = \frac{dv}{dt}\boldsymbol{\tau} + v\frac{d\theta}{dt}\boldsymbol{\nu}. \tag{2.14}$$

If the path is a straight line at time t, the normal component of the acceleration equals zero, because in that case $d\theta/dt$ is zero.

The tangential component of the acceleration arises from the rate of change of the magnitude of the velocity. The normal component of the acceleration arises from the rate of change in the direction of the velocity vector.

Figure 2.8 shows the positions on the path reached by P at time t, $P(t)$, and at time $t + dt$, $P(t + dt)$. If the path is curved, straight lines extended from these points $P(t)$ and $P(t + dt)$ perpendicular to the path will intersect at C as shown in Fig. 2.8. The distance ρ from the path to the point where these two lines intersect is called the *instantaneous radius of curvature* of the path.

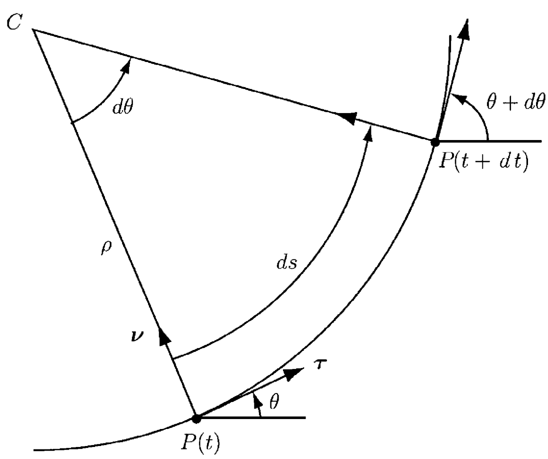

Figure 2.8

If the path is circular with radius a, then the radius of curvature equals the radius of the path, $\rho = a$. The angle $d\theta$ is the change in the path angle, and ds is the distance traveled, from t to $t + dt$. The radius of curvature ρ is related to ds by (Fig. 2.8)

$$ds = \rho d\theta.$$

Dividing by dt, one can obtain

$$\frac{ds}{dt} = v = \rho \frac{d\theta}{dt}.$$

Using this relation, one can write Eq. (2.14) as

$$\mathbf{a} = \frac{dv}{dt}\boldsymbol{\tau} + \frac{v^2}{\rho}\boldsymbol{\nu}.$$

For a given value of v, the normal component of the acceleration depends on the instantaneous radius of curvature. The greater the curvature of the path, the greater the normal component of the acceleration. When the acceleration is expressed in this way, the normal unit vector $\boldsymbol{\nu}$ must be defined to point toward the concave side of the path (Fig. 2.9). The velocity and acceleration in terms of normal and tangential components are (Fig. 2.10)

$$\mathbf{v} = v\boldsymbol{\tau} = \frac{ds}{dt}\boldsymbol{\tau}, \tag{2.15}$$

$$\mathbf{a} = a_t\boldsymbol{\tau} + a_n\boldsymbol{\nu}, \tag{2.16}$$

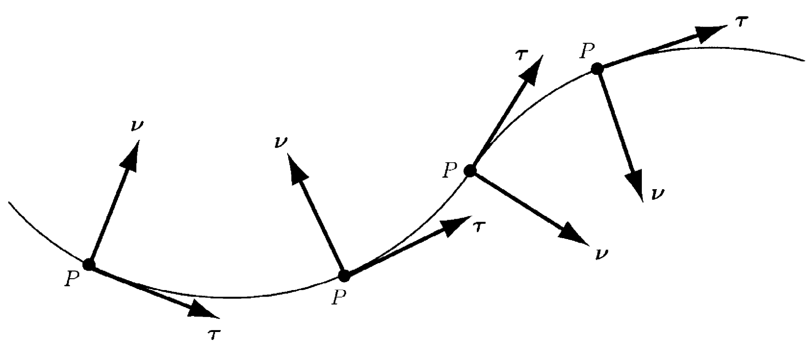

Figure 2.9

where

$$a_t = \frac{dv}{dt}, \quad a_n = v\frac{d\theta}{dt} = \frac{v^2}{\rho}. \tag{2.17}$$

If the motion occurs in the x–y plane of a cartesian reference frame (Fig. 2.11), and θ is the angle between the x axis and the unit vector $\boldsymbol{\tau}$, the unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are related to the cartesian unit vectors by

$$\begin{aligned} \boldsymbol{\tau} &= \cos\theta\,\mathbf{\imath} + \sin\theta\,\mathbf{\jmath} \\ \boldsymbol{\nu} &= -\sin\theta\,\mathbf{\imath} + \cos\theta\,\mathbf{\jmath}. \end{aligned} \tag{2.18}$$

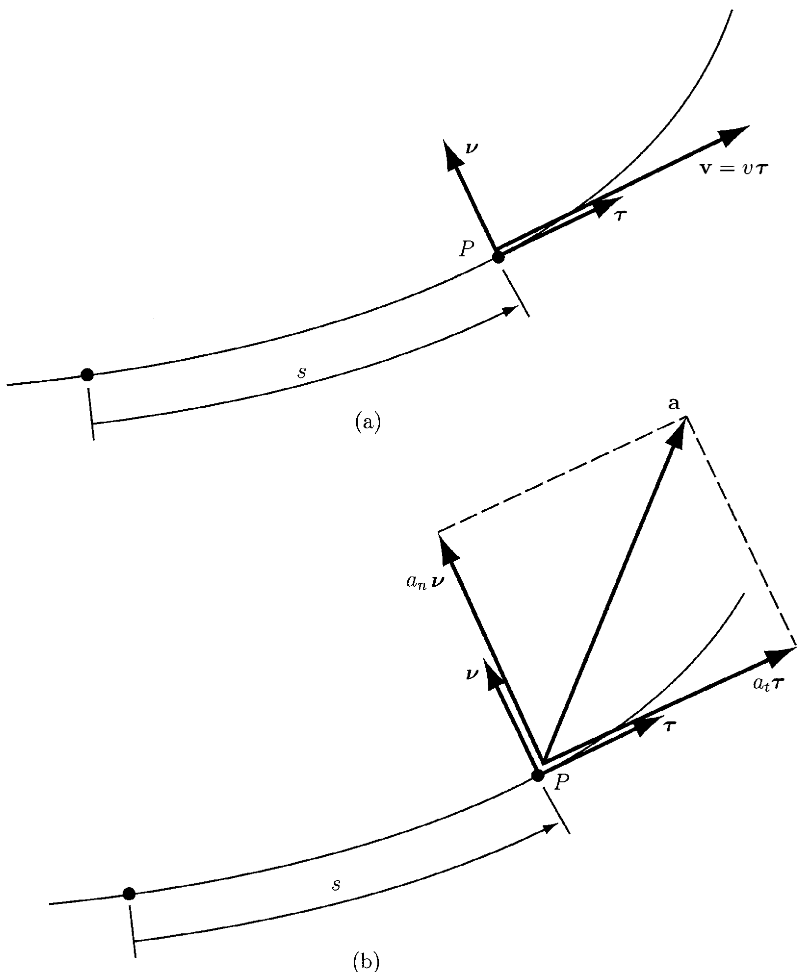

Figure 2.10

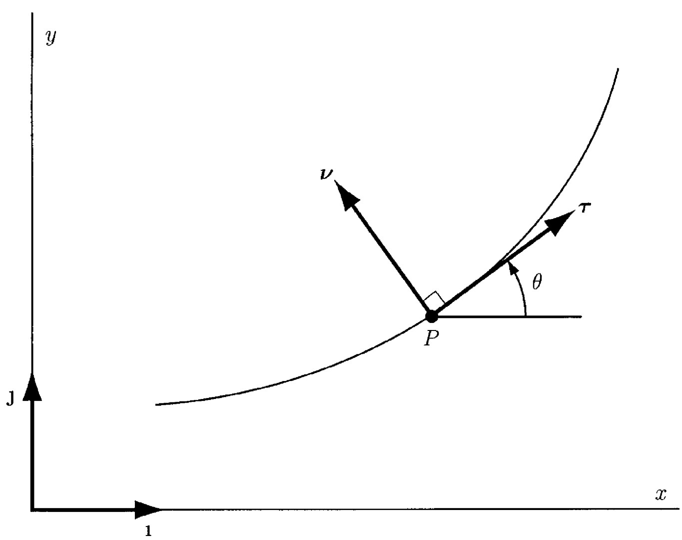

Figure 2.11

If the path in the x–y plane is described by a function $y = y(x)$, it can be shown that the instantaneous radius of curvature is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}. \tag{2.19}$$

2.6.2 CIRCULAR MOTION

The point P moves in a plane circular path of radius R as shown in Fig. 2.12. The distance s is

$$s = R\theta, \tag{2.20}$$

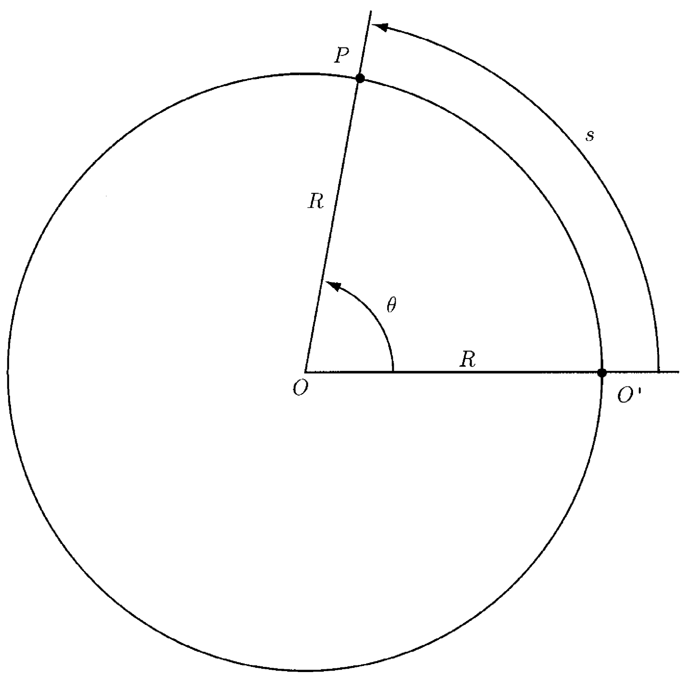

Figure 2.12

where the angle θ specifies the position of the point P along the circular path. The velocity is obtained taking the time derivative of Eq. (2.20),

$$v = \dot{s} = R\dot{\theta} = R\omega, \tag{2.21}$$

where $\omega = \dot{\theta}$ is the angular velocity of the line from the center of the path O to the point P. The tangential component of the acceleration is $a_t = dv/dt$, and

$$a_t = \dot{v} = R\dot{\omega} = R\alpha, \tag{2.22}$$

where $\alpha = \dot{\omega}$ is the angular acceleration. The normal component of the acceleration is

$$a_n = \frac{v^2}{R} = R\omega^2. \tag{2.23}$$

For the circular path the instantaneous radius of curvature is $\rho = R$.

2.6.3 SPATIAL MOTION (FRENET'S FORMULAS)

The motion of a point P along a three-dimensional path is considered (Fig. 2.13a). The tangent direction is defined by the unit tangent vector $\boldsymbol{\tau}(|\boldsymbol{\tau}| = 1)$

$$\boldsymbol{\tau} = \frac{d\mathbf{r}}{ds}. \tag{2.24}$$

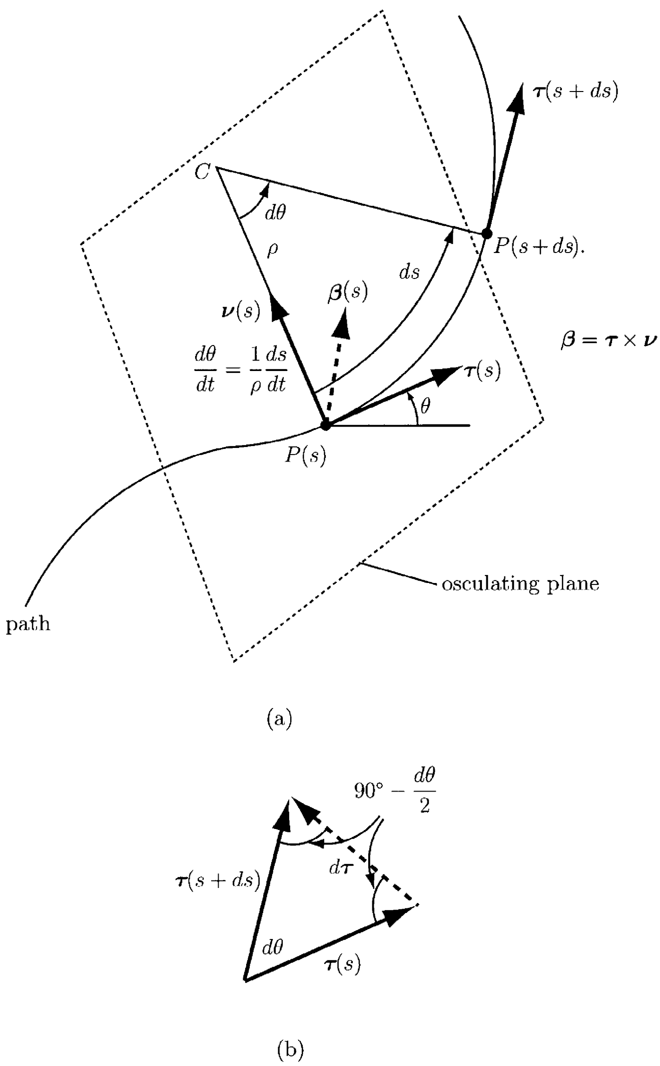

Figure 2.13

The second unit vector is derived by considering the dependence of $\boldsymbol{\tau}$ on s, $\boldsymbol{\tau} = \boldsymbol{\tau}(s)$. The dot product $\boldsymbol{\tau} \cdot \boldsymbol{\tau}$ gives the magnitude of the unit vector $\boldsymbol{\tau}$, that is,

$$\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 1. \tag{2.25}$$

Equation (2.25) can be differentiated with respect to the path variable s:

$$\frac{d\boldsymbol{\tau}}{ds} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{ds} = 0 \Rightarrow \boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{ds} = 0. \tag{2.26}$$

Equation (2.26) means that the vector $d\boldsymbol{\tau}/ds$ is always perpendicular to the vector $\boldsymbol{\tau}$. The normal direction, with the unit vector is $\boldsymbol{\nu}$, is defined to be parallel to the derivative $d\boldsymbol{\tau}/ds$. Because parallelism of two vectors corresponds to their proportionality, the normal unit vector may be written as

$$\boldsymbol{\nu} = \rho \frac{d\boldsymbol{\tau}}{ds}, \tag{2.27}$$

or

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{1}{\rho} \boldsymbol{\nu}, \tag{2.28}$$

where ρ is the radius of curvature.

Figure 2.13a depicts the tangent and normal vectors associated with two points, $P(s)$ and $P(s + ds)$. The two points are separated by an infinitesimal distance ds measured along an arbitrary planar path. The point C is the intersection of the normal vectors at the two positions along the curve, and it is the center of curvature. Because ds is infinitesimal, the arc $P(s)P(s + ds)$ seems to be circular. The radius ρ of this arc is the radius of curvature. The formula for the arc of a circle is

$$d\theta = ds/\rho.$$

The angle $d\theta$ between the normal vectors in Fig. 2.13a is also the angle between the tangent vectors $\boldsymbol{\tau}(s + ds)$ and $\boldsymbol{\tau}(s)$. The vector triangle $\boldsymbol{\tau}(s + ds)$, $\boldsymbol{\tau}(s)$, $d\boldsymbol{\tau} = \boldsymbol{\tau}(s + ds) - \boldsymbol{\tau}(s)$ in Fig. 2.13b is isosceles because $|\boldsymbol{\tau}(s + ds)| = |\boldsymbol{\tau}(s)| = 1$. Hence, the angle between $d\boldsymbol{\tau}$ and either tangent vector is $90^\circ - d\theta/2$. Since $d\theta$ is infinitesimal, the vector $d\boldsymbol{\tau}$ is perpendicular to the vector $\boldsymbol{\tau}$ in the direction of $\boldsymbol{\nu}$. A unit vector has a length of 1, so

$$|d\boldsymbol{\tau}| = d\theta |\boldsymbol{\tau}| = \frac{ds}{\rho}.$$

Any vector may be expressed as the product of its magnitude and a unit vector defining the sense of the vector

$$d\boldsymbol{\tau} = |d\boldsymbol{\tau}| \boldsymbol{\nu} = \frac{ds}{\rho} \boldsymbol{\nu}. \tag{2.29}$$

Note that the radius of curvature ρ is generally not a constant.

The tangent ($\boldsymbol{\tau}$) and normal ($\boldsymbol{\nu}$) unit vectors at a selected position form a plane, the *osculating plane*, that is tangent to the curve. Any plane containing $\boldsymbol{\tau}$ is tangent to the curve. When the path is not planar, the orientation of the oscillating plane containing the $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ pair will depend on the position along the curve. The direction perpendicular to the osculating plane is called the *binormal*, and the corresponding unit vector is $\boldsymbol{\beta}$. The cross product of two unit vectors is a unit vector perpendicular to the original two, so the binormal direction may be defined such that

$$\boldsymbol{\beta} + \boldsymbol{\tau} \times \boldsymbol{\nu}.$$

Next the derivative of the $\boldsymbol{\nu}$ unit vector with respect to s in terms of its tangent, normal, and binormal components will be calculated. The component of any vector in a specific direction may be obtained from a dot product with a unit vector in that direction:

$$\frac{d\boldsymbol{\nu}}{ds} = \left(\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\nu}}{ds}\right)\boldsymbol{\tau} + \left(\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\nu}}{ds}\right) + \left(\boldsymbol{\beta} \cdot \frac{d\boldsymbol{\nu}}{ds}\right)\boldsymbol{\beta}. \tag{2.31}$$

The orthogonality of the unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, $\boldsymbol{\tau} \perp \boldsymbol{\nu}$, requires that

$$\boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0. \tag{2.32}$$

Equation (2.32) can be differentiated with respect to the path variable s:

$$\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\nu}}{ds} + \boldsymbol{\nu} \cdot \frac{d\boldsymbol{\tau}}{ds} = 0$$

or

$$\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\nu}}{ds} = -\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\tau}}{ds} = -\boldsymbol{\nu} \cdot \left(\frac{1}{\rho}\boldsymbol{\nu}\right) = -\frac{1}{\rho}. \tag{2.33}$$

Because $\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1$, one may find that

$$\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\nu}}{ds} = 0. \tag{2.34}$$

The derivative of the binormal component is

$$\frac{1}{T} = \boldsymbol{\beta} \cdot \frac{d\boldsymbol{\nu}}{ds}, \tag{2.35}$$

or

$$\frac{d\boldsymbol{\nu}}{ds} = \frac{1}{T}\boldsymbol{\beta}, \tag{2.36}$$

where T is the *torsion*. The reciprocal is used for consistency with Eq. (2.28). The torsion T has the dimension of length.

Substitution of Eqs. (2.33), (2.34), and (2.35) into Eq. (2.31) results in

$$\frac{d\boldsymbol{\nu}}{ds} = -\frac{1}{\rho}\boldsymbol{\tau} + \frac{1}{T}\boldsymbol{\beta}. \tag{2.37}$$

The derivative of $\boldsymbol{\beta}$,

$$\frac{d\boldsymbol{\beta}}{ds} = \left(\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\beta}}{ds}\right)\boldsymbol{\tau} + \left(\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\nu}}{ds}\right)\boldsymbol{\nu} + \left(\boldsymbol{\beta} \cdot \frac{d\boldsymbol{\beta}}{ds}\right)\boldsymbol{\beta}, \tag{2.38}$$

may be obtained by a similar approach.

Using the fact that $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ are mutually orthogonal, and Eqs. (2.28) and (2.37), yields

$$
\begin{aligned}
\boldsymbol{\tau}\cdot\boldsymbol{\beta} = 0 &\Rightarrow \boldsymbol{\tau}\cdot\frac{d\boldsymbol{\beta}}{ds} = -\frac{d\boldsymbol{\tau}}{ds}\cdot\boldsymbol{\beta} = -\frac{1}{\rho}\boldsymbol{\nu}\cdot\boldsymbol{\beta} = 0\\
\boldsymbol{\nu}\cdot\boldsymbol{\beta} = 0 &\Rightarrow \boldsymbol{\nu}\cdot\frac{d\boldsymbol{\beta}}{ds} = -\frac{d\boldsymbol{\nu}}{ds}\cdot\boldsymbol{\beta} = -\frac{1}{T}\\
\boldsymbol{\beta}\cdot\boldsymbol{\beta} = 1 &\Rightarrow \boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{ds} = 0.
\end{aligned}
\tag{2.39}
$$

The result is

$$\frac{d\boldsymbol{\beta}}{ds} = -\frac{1}{T}\boldsymbol{\nu}. \tag{2.40}$$

Because $\boldsymbol{\nu}$ is a unit vector, this relation provides an alternative to Eq. (2.36) for the torsion:

$$\frac{1}{T} = -\left|\frac{d\boldsymbol{\beta}}{ds}\right|. \tag{2.41}$$

Equations (2.28), (2.37), and (2.40) are the Frenet's formulas for a spatial curve.

Next the path is given in parametric form, the x, y, and z coordinates are given in terms of a parameter α. The position vector may be written as

$$\mathbf{r} = x(\alpha)\boldsymbol{\imath} + y(\alpha)\boldsymbol{\jmath} + z(\alpha)\mathbf{k}. \tag{2.42}$$

The unit tangent vector is

$$\boldsymbol{\tau} = \frac{d\mathbf{r}}{d\alpha}\frac{d\alpha}{ds} = \frac{\mathbf{r}'(\alpha)}{s'(\alpha)}, \tag{2.43}$$

where a prime denotes differentiation with respect to α and

$$\mathbf{r}' = x'\boldsymbol{\imath} + y'\boldsymbol{\jmath} + z'\mathbf{k}.$$

Using the fact that $|\boldsymbol{\tau}| = 1$, one may write

$$s' = (\mathbf{r}'\cdot\mathbf{r}')^{1/2} = [(x')^2 + (y')^2 + (z')^2]^{1/2}. \tag{2.44}$$

The arc length s may be computed with the relation

$$s = \int_{\alpha_o}^{\alpha} [(x')^2 + (y')^2 + (z')^2]^{1/2}\, d\alpha, \tag{2.45}$$

where α_o is the value at the starting position. The value of s' found from Eq. (2.44) may be substituted into Eq. (2.43) to calculate the tangent vector

$$\boldsymbol{\tau} = \frac{x'\boldsymbol{\imath} + y'\boldsymbol{\jmath} + z'\mathbf{k}}{\sqrt{(x')^2 + (y')^2 + (z')^2}}. \tag{2.46}$$

From Eqs. (2.43) and (2.28), the normal vector is

$$\boldsymbol{\nu} = \rho\frac{d\boldsymbol{\tau}}{ds} = \rho\frac{d\boldsymbol{\tau}}{d\alpha}\frac{d\alpha}{ds} = \frac{\rho}{s'}\left(\frac{\mathbf{r}''}{s'} - \frac{\mathbf{r}'s''}{(s')^2}\right) = \frac{\rho}{(s')^3}(\mathbf{r}''s' - \mathbf{r}'s''). \tag{2.47}$$

The value of s' is given by Eq. (2.44) and the value of s'' is obtained differentiating Eq. (2.44):

$$s'' = \frac{\mathbf{r}' \cdot \mathbf{r}''}{(\mathbf{r}' \cdot \mathbf{r}')^{1/2}} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{s'}. \tag{2.48}$$

The expression for the normal vector is obtained by substituting Eq. (2.48) into Eq. (2.47):

$$\boldsymbol{\nu} = \frac{\rho}{(s')^4}[\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')]. \tag{2.49}$$

Because $\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1$, the radius of curvature is

$$\begin{aligned} \frac{1}{\rho} &= \frac{\rho}{(s')^4}|[\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')]| \\ &= \frac{\rho}{(s')^4}[\mathbf{r}'' \cdot \mathbf{r}''(s')^4 - 2(\mathbf{r}' \cdot \mathbf{r}'')^2(s')^2 + \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')]^{1/2}, \end{aligned}$$

which simplifies to

$$\frac{1}{\rho} = \frac{1}{(s')^3}[\mathbf{r}'' \cdot \mathbf{r}''(s')^2 - (\mathbf{r}' \cdot \mathbf{r}'')^2]^{1/2}. \tag{2.50}$$

In the case of a planar curve $y = y(x)$ $(\alpha = x)$, Eq. (2.50) reduces to Eq. (2.19).

The binomial vector may be calculated with the relation

$$\begin{aligned} \boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu} &= \frac{\mathbf{r}'}{s'} \times \frac{\rho}{(s')^4}[\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')] \\ &= \frac{\rho}{(s')^3}\mathbf{r}' \times \mathbf{r}''. \end{aligned} \tag{2.51}$$

The result of differentiating Eq. (2.51) may be written as

$$\frac{d\boldsymbol{\beta}}{ds} = \frac{1}{s'}\frac{d\boldsymbol{\beta}}{d\alpha} = \frac{1}{s'}\frac{d}{d\alpha}\left[\frac{\rho}{(s')^3}\right](\mathbf{r}' \times \mathbf{r}'') + \frac{\rho}{(s')^4}(\mathbf{r}' \times \mathbf{r}'''). \tag{2.52}$$

The torsion T may be obtained by applying the formula

$$\begin{aligned} \frac{1}{T} &= -\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\beta}}{ds} \\ &= -\frac{\rho}{(s')^4}[\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')] \cdot \left[\frac{1}{s'}\frac{d}{d\alpha}\left(\frac{\rho}{(s')^3}\right)(\mathbf{r}' \times \mathbf{r}'') + \frac{\rho}{(s')^4}(\mathbf{r}' \times \mathbf{r}''')\right]. \end{aligned}$$

The preceding equation may be simplified and T can be calculated from

$$\frac{1}{T} = -\frac{\rho^2}{(s')^6}[\mathbf{r}'' \cdot (\mathbf{r}' \times \mathbf{r}''')]. \tag{2.53}$$

The expressions for the velocity and acceleration in normal and tangential directions for three-dimensional motions are identical in form to the expressions for planar motion. The velocity is a vector whose magnitude equals the rate of change of distance, and whose direction is tangent to the path. The acceleration has a component tangential to the path equal to the rate of change of the magnitude of the velocity, and a component perpendicular to the path that depends on the magnitude of the velocity and the instantaneous radius of curvature of the path. In planar motion, the normal unit vector $\boldsymbol{\nu}$ is parallel to the plane of motion. In three-dimensional motion, $\boldsymbol{\nu}$ is parallel to the osculating plane, whose orientation depends on the nature of the path. The binomial vector $\boldsymbol{\beta}$ is a unit vector that is perpendicular to the osculating plane and therefore defines its orientation.

2.6.4 POLAR COORDINATES

A point P is considered in the x–y plane of a cartesian coordinate system. The position of the point P relative to the origin O may be specified either by its cartesian coordinates x, y or by its polar coordinates r, θ (Fig. 2.14). The polar coordinates are defined by:

- The unit vector $\mathbf{u}_r$, which points in the direction of the radial line from the origin O to the point P
- The unit vector $\mathbf{u}_\theta$, which is perpendicular to $\mathbf{u}_r$ and points in the direction of increasing the angle θ

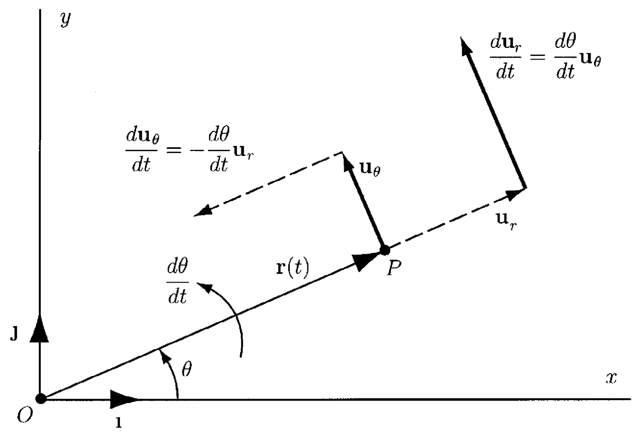

Figure 2.14

The unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are related to the cartesian unit vectors $\mathbf{\imath}$ and $\mathbf{\jmath}$ by

$$\begin{aligned} \mathbf{u}_r &= \cos\theta\mathbf{\imath} + \sin\theta\mathbf{\jmath}, \\ \mathbf{u}_\theta &= -\sin\theta\mathbf{\imath} + \cos\theta\mathbf{\jmath}. \end{aligned} \tag{2.54}$$

The position vector $\mathbf{r}$ from O to P is

$$\mathbf{r} = r\mathbf{u}_r, \tag{2.55}$$

where r is the magnitude of the vector $\mathbf{r}$, $r = |\mathbf{r}|$.

The velocity of the point P in terms of polar coordinates is obtained by taking the time derivative of Eq. (2.55):

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt}. \tag{2.56}$$

The time derivative of the rotating unit vector $\mathbf{u}_r$ is

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\theta}{dt}\mathbf{u}_\theta = \omega\mathbf{u}_\theta, \tag{2.57}$$

where $\omega = d\theta/dt$ is the angular velocity.

If we substitute Eq. (2.57) into Eq. (2.56), the velocity of P is

$$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta = \frac{dr}{dt}\mathbf{u}_r + r\omega\mathbf{u}_\theta = \dot{r}\mathbf{u}_r + r\omega\mathbf{u}_\theta, \tag{2.58}$$

or

$$\mathbf{v} = v_r\mathbf{u}_r + v_\theta\mathbf{u}_\theta, \tag{2.59}$$

where

$$v_r = \frac{dr}{dt} = \dot{r} \quad \text{and} \quad v_\theta = r\omega. \tag{2.60}$$

The acceleration of the point P is obtained by taking the time derivative of Eq. (2.58):

$$\begin{aligned}\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta \\ &+ r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt}.\end{aligned} \tag{2.61}$$

As P moves, $\mathbf{u}_\theta$ also rotates with angular velocity $d\theta/dt$. The time derivative of the unit vector $\mathbf{u}_\theta$ is in the $-\mathbf{u}_r$ direction if $d\theta/dt$ is positive:

$$\frac{d\mathbf{u}_\theta}{dt} = -\frac{d\theta}{dt}\mathbf{u}_r. \tag{2.62}$$

If Eq. (2.62) and Eq. (2.57) are substituted into Eq. (2.61), the acceleration of the point P is

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta.$$

Thus, the acceleration of P is

$$\mathbf{a} = a_r\mathbf{u}_r + a_\theta\mathbf{u}_\theta, \tag{2.63}$$

where

$$\begin{aligned}a_r &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2r}{dt^2} - r\omega^2 = \ddot{r} - r\omega^2 \\ a_\theta &= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = r\alpha + 2\omega\frac{dr}{dt} = r\alpha + 2\dot{r}\omega.\end{aligned} \tag{2.64}$$

The term

$$\alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

is called the angular acceleration.

The radial component of the acceleration $-r\omega^2$ is called *the centripetal* acceleration. The transverse component of the acceleration $2\omega(dr/dt)$ is called the *Coriolis* acceleration.

2.6.5 CYLINDRICAL COORDINATES

The cylindrical coordinates r, θ, and z describe the motion of a point P in the xyz space as shown in Fig. 2.15. The cylindrical coordinates r and θ are the polar coordinates of P measured in the plane parallel to the x–y plane, and the unit vectors $\mathbf{u}_r$, and $\mathbf{u}_\theta$ are the same. The coordinate z measures the position of the point P perpendicular to the x–y plane. The unit vector $\mathbf{k}$ attached to the coordinate z points in the positive z axis direction. The position vector $\mathbf{r}$ of the point P in terms of cylindrical coordinates is

$$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}. \tag{2.65}$$

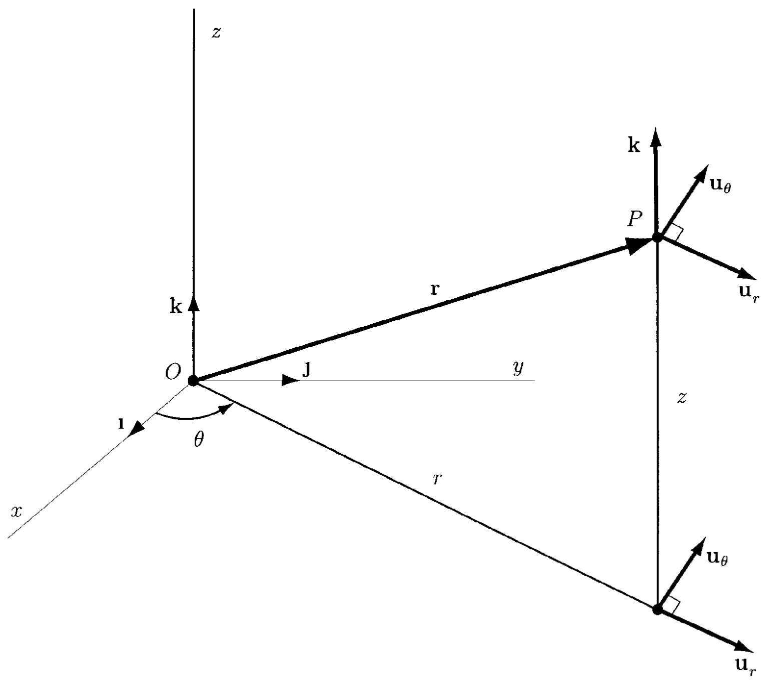

Figure 2.15

The coordinate r in Eq. (2.65) is not equal to the magnitude of $\mathbf{r}$ except when the point P moves along a path in the x–y plane.

The velocity of the point P is

$$\begin{aligned}\mathbf{v} = \frac{d\mathbf{r}}{dt} &= v_r\mathbf{u}_r + v_\theta\mathbf{u}_\theta + v_z\mathbf{k}\\ &= \frac{dr}{dt}\mathbf{u}_r + r\omega\mathbf{u}_\theta + \frac{dz}{dt}\mathbf{k}\\ &= \dot{r}\mathbf{u}_r + r\omega\mathbf{u}_\theta + \dot{z}\mathbf{k},\end{aligned} \tag{2.66}$$

and the acceleration of the point P is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_r\mathbf{u}_r + a_\theta\mathbf{u}_\theta + a_z\mathbf{k}, \tag{2.67}$$

where

$$\begin{aligned} a_r &= \frac{d^2 r}{dt^2} - r\omega^2 = \ddot{r} - r\omega^2 \\ a_\theta &= r\alpha + 2\frac{dr}{dt}\omega = r\alpha + 2\dot{r}\omega \\ a_z &= \frac{d^2 z}{dt^2} = \ddot{z}. \end{aligned} \tag{2.68}$$

2.7 Relative Motion

Suppose that A and B are two points that move relative to a reference frame with origin at point O (Fig. 2.16). Let $\mathbf{r}_A$ and $\mathbf{r}_B$ be the position vectors of points A and B relative to O. The vector $\mathbf{r}_{BA}$ is the position vector of point A relative to point B. These vectors are related by

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{BA}. \tag{2.69}$$

The time derivative of Eq. (2.69) is

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}, \tag{2.70}$$

where $\mathbf{v}_A$ is the velocity of A relative to O, $\mathbf{v}_B$ is the velocity of B relative to O, and $\mathbf{v}_{AB} = d\mathbf{r}_{AB}/dt = \dot{r}_{AB}$ is the velocity of A relative to B. The time derivative of Eq. (2.70) is

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{AB}, \tag{2.71}$$

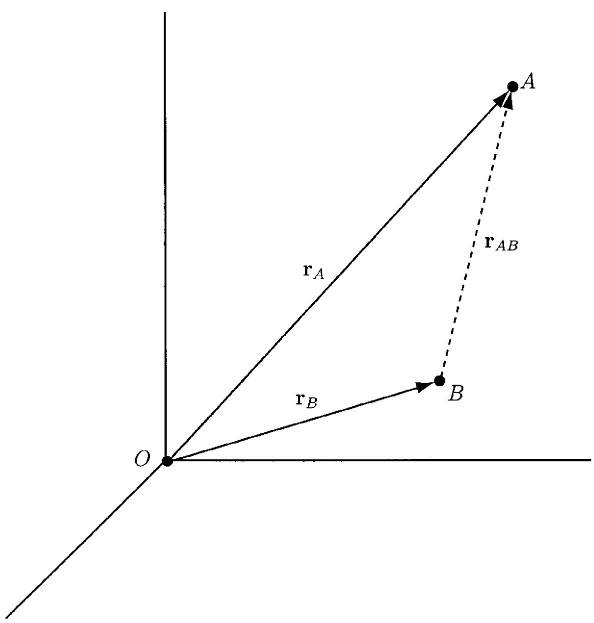

Figure 2.16

where $\mathbf{a}_A$ and $\mathbf{a}_B$ are the accelerations of A and B relative to O and $\mathbf{a}_{AB} = d\mathbf{v}_{AB}/dt = \ddot{r}_{AB}$ is the acceleration of A relative to B.

3. Dynamics of a Particle

3.1 Newton's Second Law

Classical mechanics was established by Isaac Newton with the publication of *Philosophiae naturalis principia mathematica*, in 1687. Newton stated three "laws" of motion, which may be expressed in modern terms:

1. When the sum of the forces acting on a particle is zero, its velocity is constant. In particular, if the particle is initially stationary, it will remain stationary.
2. When the sum of the forces acting on a particle is not zero, the sum of the forces is equal to the rate of change of the *linear momentum* of the particle.
3. The forces exerted by two particles on each other are equal in magnitude and opposite in direction.

The linear momentum of a particle is the product of the mass of the particle, m, and the velocity of the particle, $\mathbf{v}$. Newton's second law may be written as

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}), \tag{3.1}$$

where $\mathbf{F}$ is the total force on the particle. If the mass of the particle is constant, $m =$ constant, the total force equals the product of its mass and acceleration, $\mathbf{a}$:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}. \tag{3.2}$$

Newton's second law gives precise meanings to the terms *mass* and *force*. In SI units, the unit of mass is the kilogram (kg). The unit of force is the newton (N), which is the force required to give a mass of 1 kilogram an acceleration of 1 meter per second squared:

$$1\ \text{N} = (1\ \text{kg})(1\ \text{m/s}^2) = 1\ \text{kg m/s}^2.$$

In U.S. customary units, the unit of force is the pound (lb). The unit of mass is the slug, which is the amount of mass accelerated at 1 foot per second squared by a force of 1 pound:

$$1\ \text{lb} = (1\ \text{slug})(1\ \text{ft/s}^2),\ \text{or}\ 1\ \text{slug} = 1\ \text{lb s}^2/\text{lb}.$$

3.2 Newtonian Gravitation

Newton's postulate for the magnitude of gravitational force F between two particles in terms of their masses m_1 and m_2 and the distance r between them (Fig. 3.1) may be expressed as

$$F = \frac{Gm_1 m_2}{r^2}, \tag{3.3}$$

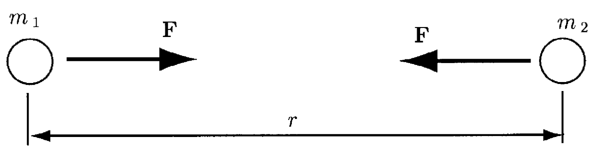

Figure 3.1

where G is called the *universal gravitational constant.* Equation (3.3) may be used to approximate the weight of a particle of mass m due to the gravitational attraction of the earth,

$$W = \frac{Gmm_E}{r^2}, \tag{3.4}$$

where m_E is the mass of the earth and r is the distance from the center of the earth to the particle. When the weight of the particle is the only force acting on it, the resulting acceleration is called the acceleration due to gravity. In this case, Newton's second law states that $W = ma$, and from Eq. (3.4) the acceleration due to gravity is

$$a = \frac{Gm_E}{r^2}. \tag{3.5}$$

The acceleration due to gravity at sea level is denoted by g. From Eq. (3.5) one may write $Gm_E = gR_E^2$, where R_E is the radius of the earth. The expression for the acceleration due to gravity at a distance r from the center of the earth in terms of the acceleration due to gravity at sea level is

$$a = g\frac{R_E^2}{r^2}. \tag{3.6}$$

At sea level, the weight of a particle is given by

$$W = mg. \tag{3.7}$$

The value of g varies on the surface of the earth from one location to another. The values of g used in examples and problems are $g = 9.81$ m/s^2 in SI units and $g = 32.2$ ft/s^2 in U.S. customary units.

3.3 Inertial Reference Frames

Newton's laws do not give accurate results if a problem involves velocities that are not small compared to the velocity of light (3×10^8 m/s). Einstein's theory of relativity may be applied to such problems. Newtonian mechanics also fails in problems involving atomic dimensions. Quantum mechanics may be used to describe phenomena on the atomic scale.

The position, velocity, and acceleration of a point are specified, in general, relative to an arbitrary reference frame. Newton's second law cannot be expressed in terms of just any reference frame. Newton stated that the second law should be expressed in terms of a reference frame at rest with respect to the "fixed stars." Newton's second law may be applied with good results using reference frames that accelerate and rotate by properly accounting for the acceleration and rotation. Newton's second law, Eq. (3.2), may be expressed in terms of a reference frame that is fixed relative to the earth. Equation (3.2) may be applied using a reference that translates at constant velocity relative to the earth.

If a reference frame may be used to apply Eq. (3.2), it is said to be a *Newtonian* or *inertial* reference frame.

3.4 Cartesian Coordinates

To apply Newton's second law in a particular situation, one may choose a coordinate system. Newton's second law in a cartesian reference frame (Fig. 3.2) may be expressed as

$$\sum \mathbf{F} = m\mathbf{a}, \tag{3.8}$$

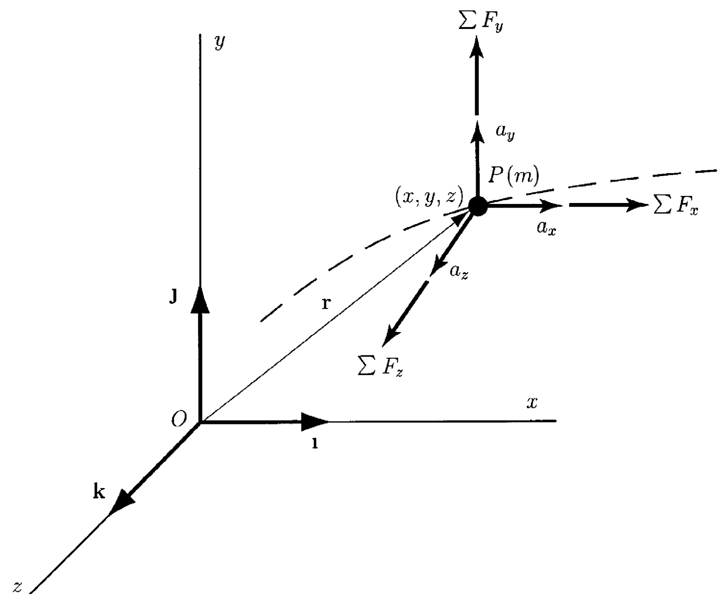

Figure 3.2

where $\sum \mathbf{F} = \sum F_x \mathbf{\imath} + \sum F_y \mathbf{\jmath} + \sum F_z \mathbf{k}$ is the sum of the forces acting on a particle P of mass m, and

$$\mathbf{a} = a_x \mathbf{\imath} + a_y \mathbf{\jmath} + a_z \mathbf{k} = \ddot{x}\mathbf{\imath} + \ddot{y}\mathbf{\jmath} + \ddot{z}\mathbf{k}$$

is the acceleration of the particle. When x, y, and z components are equated, three scalar equations of motion are obtained,

$$\sum F_x = ma_x = m\ddot{x}, \quad \sum F_y = ma_y = m\ddot{y}, \quad \sum F_z = ma_z = m\ddot{z}, \tag{3.9}$$

or the total force in each coordinate direction equals the product of the mass and component of the acceleration in that direction.

Projectile Problem

An object P, of mass m, is launched through the air (Fig. 3.3). The force on the object is just the weight of the object (the aerodynamic forces are neglected). The sum of the forces is $\sum \mathbf{F} = -mg\mathbf{\jmath}$. From Eq. (3.9) one may obtain

$$a_x = \ddot{x} = 0, \quad a_y = z\ddot{y} = -g, \quad a_z = \ddot{z} = 0.$$

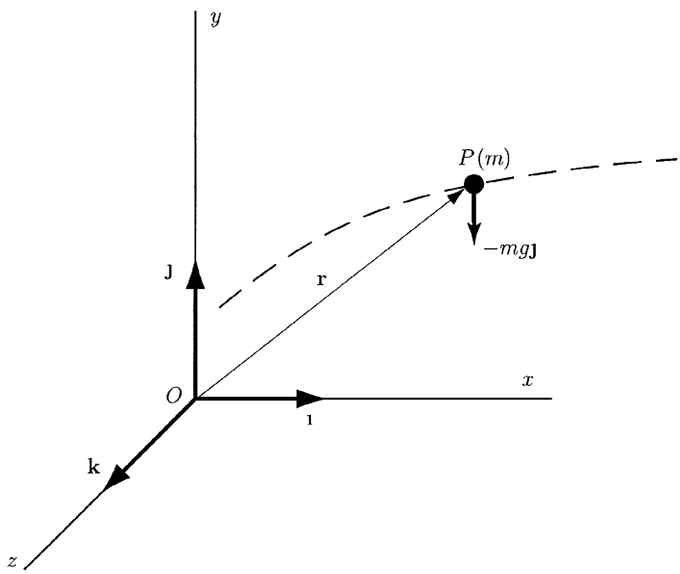

Figure 3.3

The projectile accelerates downward with the acceleration due to gravity.

Straight Line Motion

For straight line motion along the x axis, Eqs. (3.9) are

$$\sum F_x = m\ddot{x}, \quad \sum F_y = 0, \quad \sum F_z = 0.$$

3.5 Normal and Tangential Components

A particle P of mass m moves on a curved path (Fig. 3.4). One may resolve the sum of the forces $\sum \mathbf{F}$ acting on the particle into normal F_n and tangential F_t components:

$$\sum \mathbf{F} = F_t \boldsymbol{\tau} + F_n \boldsymbol{\nu}.$$

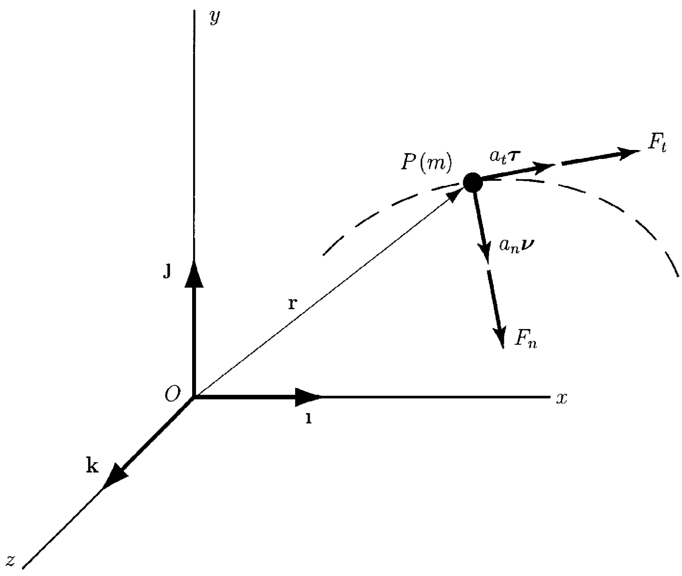

Figure 3.4

The acceleration of the particle in terms of normal and tangential components is

$$\mathbf{a} = a_t\boldsymbol{\tau} + a_n\boldsymbol{\nu}.$$

Newton's second law is

$$\begin{aligned} \sum \mathbf{F} &= m\mathbf{a} \\ F_t\boldsymbol{\tau} + F_n\boldsymbol{\nu} &= m(a_t\boldsymbol{\tau} + a_n\boldsymbol{\nu}), \end{aligned} \tag{3.10}$$

where

$$a_t = \frac{dv}{dt} = \dot{v}, \quad a_n = \frac{v^2}{\rho}.$$

When the normal and tangential components in Eq. (3.10) are equated, two scalar equations of motion are obtained:

$$F_t = m\dot{v}, \quad F_n = m\frac{v^2}{\rho}. \tag{3.11}$$

The sum of the forces in the tangential direction equals the product of the mass and the rate of change of the magnitude of the velocity, and the sum of the forces in the normal direction equals the product of the mass and the normal component of acceleration. If the path of the particle lies in a plane, the acceleration of the particle perpendicular to the plane is zero, and so the sum of the forces perpendicular to the plane is zero.

3.6 Polar and Cylindrical Coordinates

A particle P with mass m moves in a plane curved path (Fig. 3.5). The motion of the particle may be described in terms of polar coordinates. Resolving the

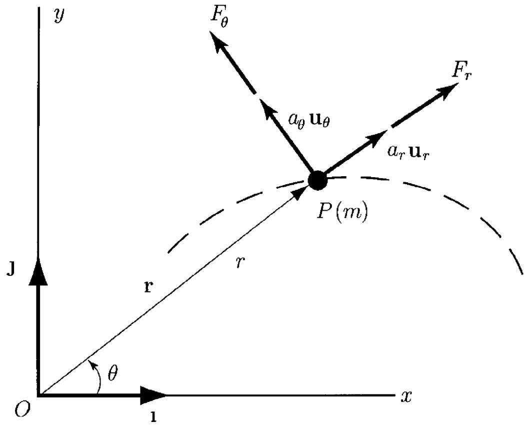

Figure 3.5

Dynamics

sum of the forces parallel to the plane into radial and transverse components gives

$$\sum \mathbf{F} = F_r \mathbf{u}_r + F_\theta \mathbf{u}_\theta,$$

and if the acceleration of the particle is expressed in terms of radial and transverse components, Newton's second law may be written in the form

$$F_r \mathbf{u}_r + F_\theta \mathbf{u}_\theta = m(a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta), \tag{3.12}$$

where

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\omega^2$$
$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = r\alpha + 2\dot{r}\omega.$$

Two scalar equations are obtained:

$$\begin{aligned} F_r &= m(\ddot{r} - r\omega^2) \\ F_\theta &= m(r\alpha + 2\dot{r}\omega). \end{aligned} \tag{3.13}$$

The sum of the forces in the radial direction equals the product of the mass and the radial component of the acceleration, and the sum of the forces in the transverse direction equals the product of the mass and the transverse component of the acceleration.

The three-dimensional motion of the particle P may be obtained using cylindrical coordinates (Fig. 3.6). The position of P perpendicular to the x–y plane is measured by the coordinate z and the unit vector $\mathbf{k}$. The sum of the forces is resolved into radial, transverse, and z components:

$$\sum \mathbf{F} = F_r \mathbf{u}_r + F_\theta \mathbf{u}_\theta + F_z \mathbf{k}.$$

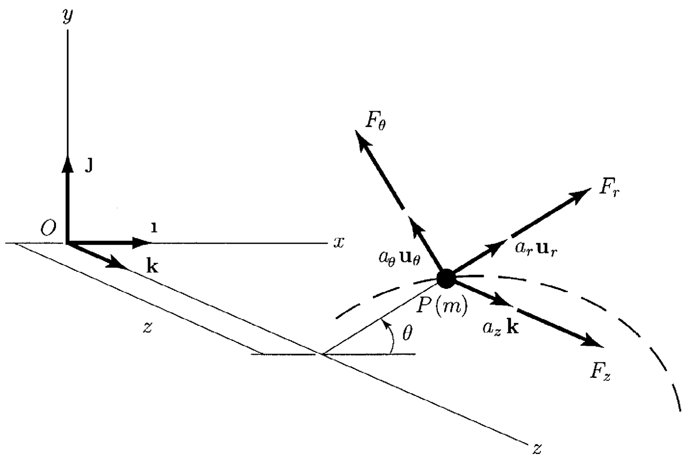

Figure 3.6

The three scalar equations of motion are the radial and transverse relations, Eq. (3.13) and the equation of motion in the z direction,

$$\begin{aligned} F_r &= m(\ddot{r} - r\omega^2) \\ F_\theta &= m(r\alpha + 2\dot{r}\omega) \\ F_z &= m\ddot{z}. \end{aligned} \tag{3.14}$$

3.7 Principle of Work and Energy

Newton's second law for a particle of mass m can be written in the form

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\dot{\mathbf{v}}. \tag{3.15}$$

The dot product of both sides of Eq. (3.15) with the velocity $\mathbf{v} = d\mathbf{r}/dt$ gives

$$\mathbf{F} \cdot \mathbf{v} = m\dot{\mathbf{v}} \cdot \mathbf{v}, \tag{3.16}$$

or

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = m\dot{\mathbf{v}} \cdot \mathbf{v}. \tag{3.17}$$

But

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \dot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v} \cdot \dot{\mathbf{v}} = 2\dot{\mathbf{v}} \cdot \mathbf{v},$$

and

$$\dot{\mathbf{v}} \cdot \mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}\frac{d}{dt}(v^2), \tag{3.18}$$

where $v^2 = \mathbf{v} \cdot \mathbf{v}$ is the square of the magnitude of $\mathbf{v}$. Using Eq. (3.18) one may write Eq. (3.17) as

$$\mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} m \, d(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2} m \, d(v^2). \tag{3.19}$$

The term

$$dU = \mathbf{F} \cdot d\mathbf{r}$$

is the *work* where $\mathbf{F}$ is the total external force acting on the particle of mass m and $d\mathbf{r}$ is the infinitesimal displacement of the particle. Integrating Eq. (3.19), one may obtain

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{v_1^2}^{v_2^2} \frac{1}{2} m \, d(v^2) = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2, \tag{3.20}$$

where v_1 and v_2 are the magnitudes of the velocity at the positions $\mathbf{r}_1$ and $\mathbf{r}_2$.

The *kinetic energy* of a particle of mass m with the velocity $\mathbf{v}$ is the term

$$T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m v^2, \tag{3.21}$$

where $|\mathbf{v}| = v$. The work done as the particle moves from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ is

$$U_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}. \tag{3.22}$$

The *principle of work and energy* may be expressed as

$$U_{12} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2. \tag{3.23}$$

The work done on a particle as it moves between two positions equals the change in its kinetic energy.

The dimensions of work, and therefore the dimensions of kinetic energy, are (force) × (length). In U.S. customary units, work is expressed in ft lb. In SI units, work is expressed in N m, or joules (J).

One may use the principle of work and energy on a system if no net work is done by internal forces. The internal friction forces may do net work on a system.

3.8 Work and Power

The position of a particle P of mass m in curvilinear motion is specified by the coordinate s measured along its path from a reference point O (Fig. 3.7a). The velocity of the particle is

$$\mathbf{v} = \frac{ds}{dt} \boldsymbol{\tau} = \dot{s} \boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ is the tangential unit vector.

Using the relation $\dot{\mathbf{v}} = d\mathbf{r}/dt$, the infinitesimal displacement $d\mathbf{r}$ along the path is

$$d\mathbf{r} = \mathbf{v} \, dt = \frac{ds}{dt} \boldsymbol{\tau} \, dt = ds \, \boldsymbol{\tau}.$$

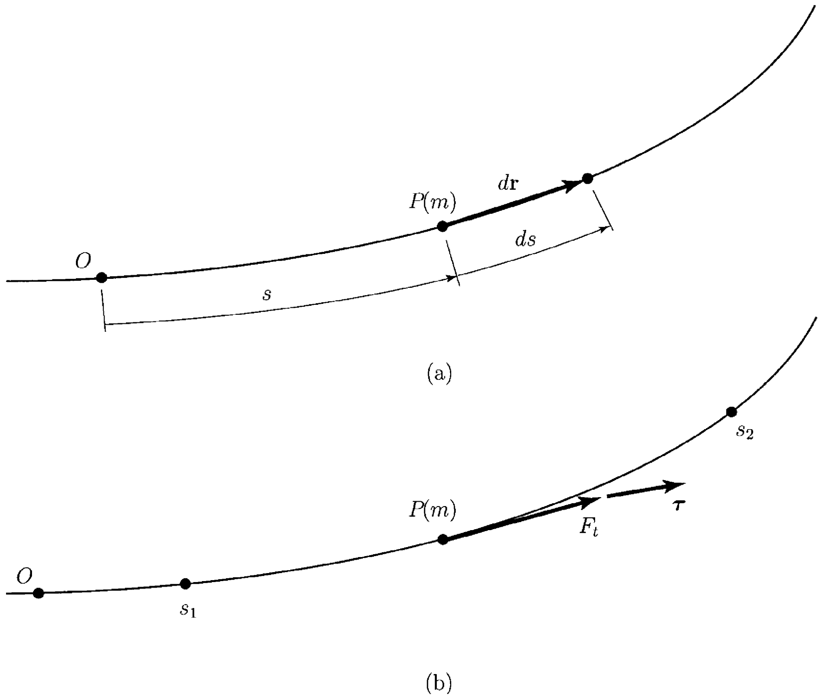

Figure 3.7

The work done by the external forces acting on the particle as result of the displacement $d\mathbf{r}$ is

$$\mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot ds\,\boldsymbol{\tau} = \mathbf{F} \cdot \boldsymbol{\tau}\, ds = F_t ds,$$

where $F_t = \mathbf{F} \cdot \boldsymbol{\tau}$ is the tangential component of the total force.

The work as the particle moves from a position s_1 to a position s_2 is (Fig. 3.7b)

$$U_{12} = \int_{s_1}^{s_2} F_t\, ds. \tag{3.24}$$

The work is equal to the integral of the tangential component of the total force with respect to distance along the path. Components of force perpendicular to the path do not do any work.

The work done by the external forces acting on a particle during an infinitesimal displacement $d\mathbf{r}$ is

$$dU = \mathbf{F} \cdot d\mathbf{r}.$$

The *power*, P, is the rate at which work is done. The power P is obtained by dividing the expression of the work by the interval of time dt during which the displacement takes place:

$$P = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v}.$$

In SI units, the power is expressed in newton meters per second, which is joules per second (J/s) or watts (W). In U.S. customary units, power is expressed in foot pounds per second or in horsepower (hp), which is 746 W or approximately 550 ft lb/s.

The power is also the rate of change of the kinetic energy of the object,

$$P = \frac{d}{dt}\left(\frac{1}{2}mv^2\right).$$

3.8.1 WORK DONE ON A PARTICLE BY A LINEAR SPRING

A linear spring connects a particle P of mass m to a fixed support (Fig. 3.8). The force exerted on the particle is

$$\mathbf{F} = -k(r - r_0)\mathbf{u}_r,$$

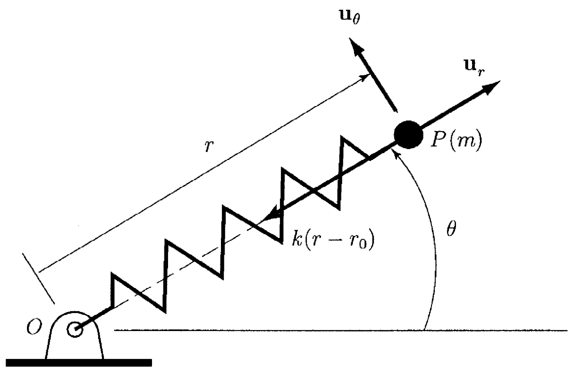

Figure 3.8

where k is the spring constant, r_0 is the unstretched length of the spring, and $\mathbf{u}_r$ is the polar unit vector. If we use the expression for the velocity in polar coordinates, the vector $d\mathbf{r} = \mathbf{v}dt$ is

$$\begin{aligned} d\mathbf{r} &= \left(\frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta\right) dt = dr\,\mathbf{u}_r + rd\theta\,\mathbf{u}_\theta \\ \mathbf{F}\cdot d\mathbf{r} &= [-k(r - r_0)\mathbf{u}_r]\cdot(dr\,\mathbf{u}_r + rd\theta\,\mathbf{u}_\theta) = -k(r - r_0)\,dr. \end{aligned} \tag{3.25}$$

One may express the work done by a spring in terms of its stretch, defined by $\delta = r - r_0$. In terms of this variable, $\mathbf{F}\cdot d\mathbf{r} = -k\delta\,d\delta$, and the work is

$$U_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\delta_1}^{\delta_2} -k\delta\,d\delta = -\frac{1}{2}k(\delta_2^2 - \delta_1^2),$$

where δ_1 and δ_2 are the values of the stretch at the initial and final positions.

3.8.2 WORK DONE ON A PARTICLE BY WEIGHT

A particle P of mass m (Fig. 3.9) moves from position 1 with coordinates (x_1, y_1, z_1) to position 2 with coordinates (x_2, y_2, z_2) in a cartesian reference frame with the y axis upward. The force exerted by the weight is

$$\mathbf{F} = -mg\mathbf{\jmath}.$$

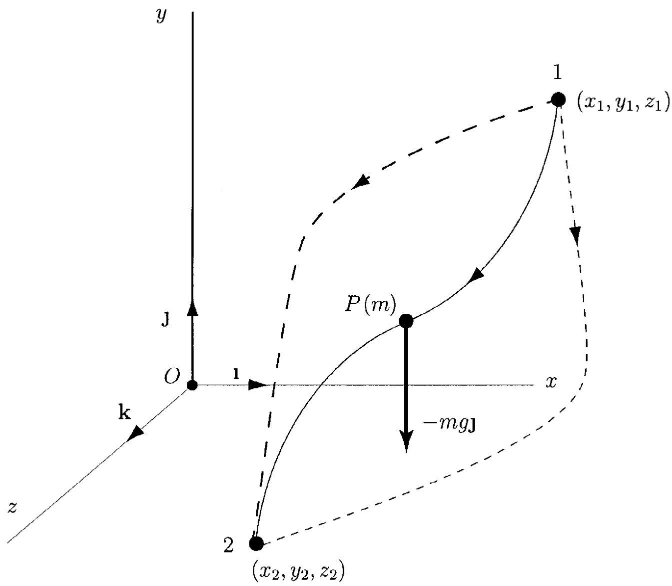

Figure 3.9

Because $\mathbf{v} = d\mathbf{r}/dt$, the expression for the vector $d\mathbf{r}$ is

$$d\mathbf{r} = \left(\frac{dx}{dt}\imath + \frac{dy}{dt}\jmath + \frac{dz}{dt}\mathbf{k}\right) dt = dx\imath + dy\jmath + dz\mathbf{k}.$$

The dot product of $\mathbf{F}$ and $d\mathbf{r}$ is

$$\mathbf{F} \cdot d\mathbf{r} = (-mg\jmath) \cdot (dx\imath + dy\jmath + dz\mathbf{k}) = -mg\ dy.$$

The work done as P moves from position 1 to position 2 is

$$U_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{y_1}^{y_2} -mg\ dy = -mg(y_2 - y_1).$$

The work is the product of the weight and the change in the height of the particle. The work done is negative if the height increases and positive if it decreases. The work done is the same no matter what path the particle follows from position 1 to position 2. To determine the work done by the weight of the particle, only the relative heights of the initial and final positions must be known.

3.9 Conservation of Energy

The change in the kinetic energy is

$$U_{12} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2. \tag{3.26}$$

A scalar function of position V called *potential energy* may be determined as

$$dV = -\mathbf{F} \cdot d\mathbf{r}. \tag{3.27}$$

If we use the function V, the integral defining the work is

$$U_{12} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = \int_{V_1}^{V_2} -dV = -(V_2 - V_1), \tag{3.28}$$

where V_1 and V_2 are the values of V at the positions r_1 and r_2. The principle of work and energy would then have the form

$$\frac{1}{2}mv_1^2 + V_1 = \frac{1}{2}mv_2^2 + V_2, \tag{3.29}$$

which means that the sum of the kinetic energy and the potential energy V is constant:

$$\frac{1}{2}mv^2 + V = \text{constant} \tag{3.30}$$

or

$$E = T + V = \text{constant}. \tag{3.31}$$

If a potential energy V exists for a given force $\mathbf{F}$, i.e., a function of position V exists such that $dV = -\mathbf{F} \cdot d\mathbf{r}$, then $\mathbf{F}$ is said to be *conservative*.

If all the forces that do work on a system are conservative, the total energy—the sum of the kinetic energy and the potential energies of the forces—is constant, or conseved. The system is said to be conservative.

3.10 Conservative Forces

A particle moves from position 1 to position 2. Equation (3.28) states that the work depends only on the values of the potential energy at positions 1 and 2. The work done by a conservative force as a particle moves from position 1 to position 2 is independent of the path of the particle.

A particle P of mass m slides with friction along a path of length L. The magnitude of the friction force is μmg and is opposite to the direction of the motion of the particle. The coefficient of friction is μ. The work done by the friction force is

$$U_{12} = \int_0^L -\mu mg \, ds = -\mu mgL.$$

The work is proportional to the length L of the path and therefore is not independent of the path of the particle. Friction forces are not conservative.

3.10.1 POTENTIAL ENERGY OF A FORCE EXERTED BY A SPRING

The force exerted by a linear spring attached to a fixed support is a conservative force.

In terms of polar coordinates, the force exerted on a particle (Fig. 3.8) by a linear spring is $\mathbf{F} = -k(r - r_0)\mathbf{u}_r$. The potential energy must satisfy

$$dV = -\mathbf{F} \cdot d\mathbf{r} = k(r - r_0) \, dr,$$

or

$$dV = k\delta\, d\delta,$$

where $\delta = r - r_0$ is the stretch of the spring. Integrating this equation, the potential energy of a linear spring is

$$V = \frac{1}{2} k\delta^2. \tag{3.32}$$

3.10.2 POTENTIAL ENERGY OF WEIGHT

The weight of a particle is a conservative force. The weight of the particle P of mass m (Fig. 3.9) is $F = -mg\mathbf{j}$. The potential energy V must satisfy the relation

$$dV = -\mathbf{F} \cdot d\mathbf{r} = (mg\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) = mg\, dy, \tag{3.33}$$

or

$$\frac{dV}{dy} = mg.$$

After integration of this equation, the potential energy is

$$V = mgy + C,$$

where C is an integration constant. The constant C is arbitrary, because this expression satisfies Eq. (3.33) for any value of C. For $C = 0$ the potential energy of the weight of a particle is

$$V = mgy. \tag{3.34}$$

The potential energy V is a function of position and may be expressed in terms of a cartesian reference frame as $V = V(x, y, z)$. The differential of dV is

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \tag{3.35}$$

The potential energy V satisfies the relation

$$\begin{aligned} dV &= -\mathbf{F} \cdot d\mathbf{r} = -(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= -(F_x\, dx + F_y\, dy + F_z\, dz), \end{aligned} \tag{3.36}$$

where $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. Using Eqs. (3.35) and (3.36), one may obtain

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = -(F_x\, dx + F_y\, dy + F_z\, dz),$$

which implies that

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}. \tag{3.37}$$

Given the potential energy $V = V(x, y, z)$ expressed in cartesian coordinates, the force $\mathbf{F}$ is

$$\mathbf{F} = -\left(\frac{\partial V}{\partial x}\mathbf{\imath} + \frac{\partial V}{\partial y}\mathbf{\jmath} + \frac{\partial V}{\partial z}\mathbf{k}\right) = -\nabla V, \tag{3.38}$$

where ∇V is the *gradient* of V. The gradient expressed in cartesian coordinates is

$$\nabla = \frac{\partial}{\partial x}\mathbf{\imath} + \frac{\partial}{\partial y}\mathbf{\jmath} + \frac{\partial}{\partial z}\mathbf{k}. \tag{3.39}$$

The *curl* of a vector force $\mathbf{F}$ in cartesian coordinates is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}. \tag{3.40}$$

If a force F is conservative, its curl $\nabla \times \mathbf{F}$ is zero. The converse is also true: A force $\mathbf{F}$ is conservative if its curl is zero.

In terms of cylindrical coordinates the force $\mathbf{F}$ is

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{u}_\theta + \frac{\partial V}{\partial z}\mathbf{k}\right). \tag{3.41}$$

In terms of cylindrical coordinates, the curl of the force $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{u}_r & r\mathbf{u}_\theta & \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}. \tag{3.42}$$

3.11 Principle of Impulse and Momentum

Newton's second law,

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt},$$

is integrated with respect to time to give

$$\int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2 - m\mathbf{v}_1, \tag{3.43}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the particle P at the times t_1 and t_2.

The term $\int_{t_1}^{t_2} \mathbf{F}\, dt$ is called the *linear impulse*, and the term $m\mathbf{v}$ is called the *linear momentum.*

The principle of impulse and momentum: The impulse applied to a particle during an interval of time is equal to the change in its linear momentum (Fig. 3.10).

The dimensions of the linear impulse and linear momentum are (mass) times (length)/(time).

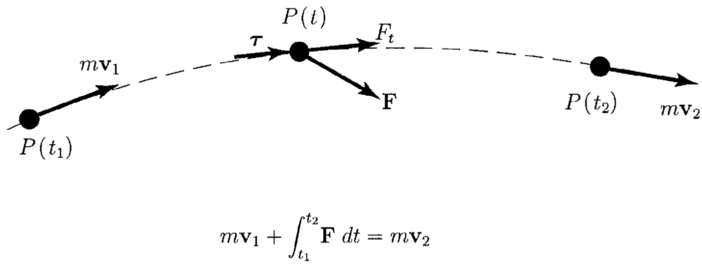

Figure 3.10

The average with respect to time of the total force acting on a particle from t_1 to t_2 is

$$\mathbf{F}_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathbf{F}\, dt,$$

so one may write Eq. (3.43) as

$$\mathbf{F}_{av}(t_2 - t_1) = m\mathbf{v}_2 - m\mathbf{v}_1. \tag{3.44}$$

An *impulsive force* is a force of relatively large magnitude that acts over a small interval of time (Fig. 3.11).

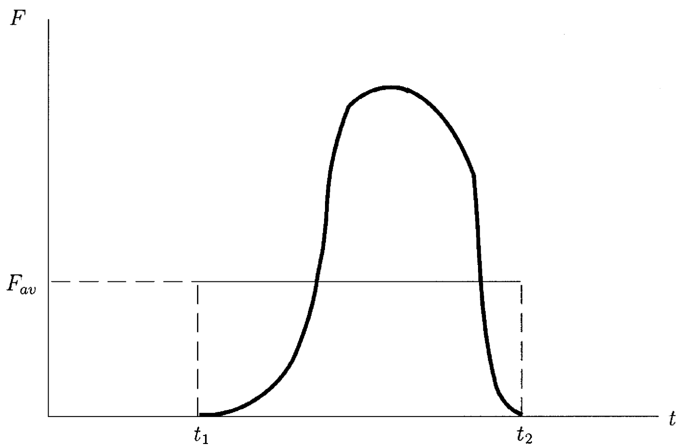

Figure 3.11

Equations (3.43) and (3.44) may be expressed in scalar forms. The sum of the forces in the tangent direction $\boldsymbol{\tau}$ to the path of the particle equals the product of its mass m and the rate of change of its velocity along the path:

$$F_t = ma_t = m\frac{dv}{dt}.$$

Integrating this equation with respect to time, one may obtain

$$\int_{t_1}^{t_2} F_t\, dt = mv_2 - mv_1, \tag{3.45}$$

where v_1 and v_2 are the velocities along the path at the times t_1 and t_2. The impulse applied to an object by the sum of the forces tangent to its path during an interval of time is equal to the change in its linear momentum along the path.

3.12 Conservation of Linear Momentum

Consider the two particles P_1 of mass m_1 and P_2 of mass m_2 shown in Fig. 3.12. The vector $\mathbf{F}_{12}$ is the force exerted by P_1 on P_2, and $\mathbf{F}_{21}$ is the force exerted by P_2 on P_1. These forces could be contact forces or could be exerted by a spring connecting the particles. As a consequence of Newton's third law, these forces are equal and opposite:

$$\mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{0}. \tag{3.46}$$

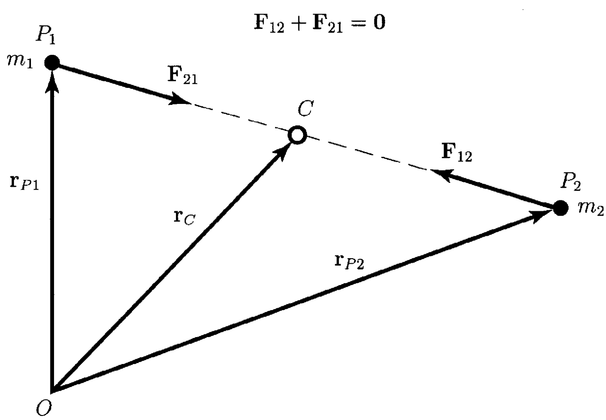

Figure 3.12

Consider that no external forces act on P_1 and P_2, or the external forces are negligible. The principle of impulse and momentum to each particle for arbitrary times t_1 and t_2 gives

$$\int_{t_1}^{t_2} \mathbf{F}_{21}\,dt = m_1\mathbf{v}_{P1}(t_2) - m_1\mathbf{v}_{P1}(t_1)$$
$$\int_{t_1}^{t_2} \mathbf{F}_{12}\,dt = m_2\mathbf{v}_{P2}(t_2) - m_1\mathbf{v}_{P2}(t_1),$$

where $\mathbf{v}_{P1}(t_1)$, $\mathbf{v}_{P1}(t_2)$ are the velocities of P_1 at the times t_1, t_2, and $\mathbf{v}_{P2}(t_1)$, $\mathbf{v}_{P2}(t_2)$ are the velocities of P_2 at the times t_1, t_2. The sum of these equations is

$$m_1\mathbf{v}_{P1}(t_1) + m_2\mathbf{v}_{P2}(t_1) = m_1\mathbf{v}_{P1}(t_2) + m_2\mathbf{v}_{P2}(t_2),$$

or the total linear momentum of P_1 and P_2 is conserved:

$$m_1\mathbf{v}_{P1} + m_2\mathbf{v}_{P2} = \text{constant}. \tag{3.47}$$

The position of the center of mass of P_1 and P_2 is (Fig. 3.12)

$$\mathbf{r}_C = \frac{m_1\mathbf{r}_{P1} + m_2\mathbf{r}_{P2}}{m_1 + m_2},$$

where $\mathbf{r}_{P1}$ and $\mathbf{r}_{P2}$ are the position vectors of P_1 and P_2. Taking the time derivative of this equation and using Eq. (3.47) one may obtain

$$(m_1 + m_2)\mathbf{v}_C = m_1\mathbf{v}_{P1} + m_2\mathbf{v}_{P2} = \text{constant}, \tag{3.48}$$

where $\mathbf{v}_C = d\mathbf{r}_C/dt$ is the velocity of the combined center of mass. The total linear momentum of the particles is conserved and the velocity of the combined center of mass of the particles P_1 and P_2 is constant.

3.13 Impact

Two particles A and B with the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ collide. The velocities of A and B after the impact are $\mathbf{v}'_A$ and $\mathbf{v}'_B$. The effects of external forces are negligible and the total linear momentum of the particles is conserved (Fig. 3.13):

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m_A\mathbf{v}'_A + m_B\mathbf{v}'_B. \tag{3.49}$$

Furthermore, the velocity $\mathbf{v}$ of the center of mass of the particles is the same before and after the impact:

$$\mathbf{v} = \frac{m_A\mathbf{v}_A + m_B\mathbf{v}_B}{m_A + m_B}. \tag{3.50}$$

If A and B remain together after the impact, they are said to undergo a *perfectly plastic impact.* Equation (3.50) gives the velocity of the center of mass of the object they form after the impact (Fig. 3.13b).

If A and B rebound, linear momentum conservation alone does not provide enough equations to determine the velocities after the impact.

3.13.1 DIRECT CENTRAL IMPACTS

The particles A and B move along a straight line with velocities v_A and v_B before their impact (Fig. 3.14a). The magnitude of the force the particles exert on each other during the impact is R (Fig. 3.14b). The impact force is parallel to the line along which the particles travel (direct central impact). The particles continue to move along the same straight line after their impact (Fig. 3.14c). The effects of external forces during the impact are negligible, and the total linear momentum is conserved:

$$m_Av_A + m_Bv_B = m_Av'_A + m_Bv'_B. \tag{3.51}$$

Another equation is needed to determine the velocities v'_A and v'_B.

Let t_1 be the time at which A and B first come into contact (Fig. 3.14a). As a result of the impact, the objects will deform. At the time t_m the particles will have reached the maximum compression (period of compression, $t_1 < t < t_m$; Fig. 3.14b). At this time the relative velocity of the particles is zero, so they have the same velocity, v_m. The particles then begin to move apart and separate at a time t_2 (Fig. 3.14c). The second period, from the

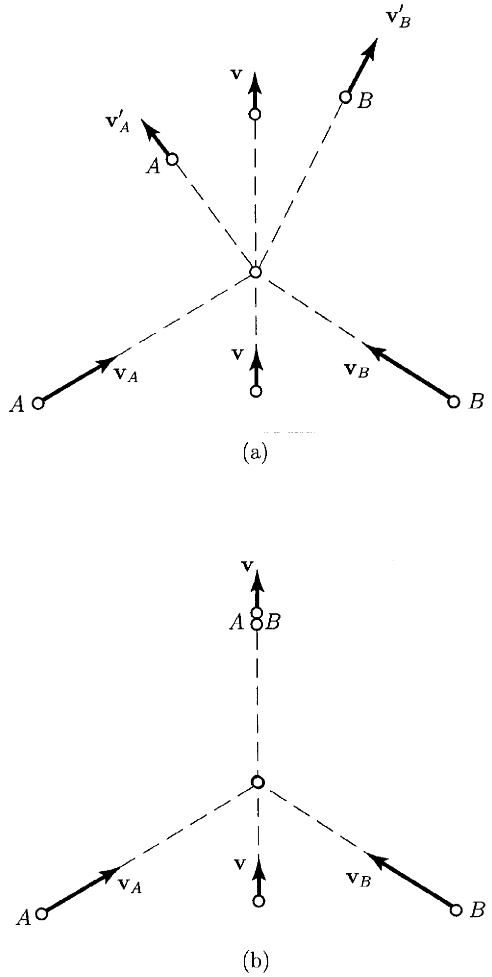

Figure 3.13

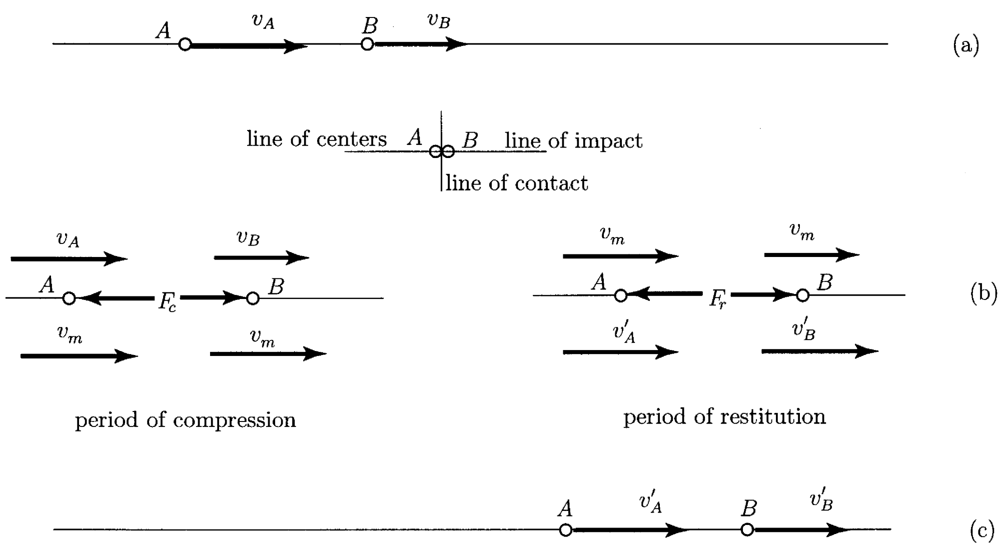

Figure 3.14

maximum compression to the instant at which the particles separate, is termed the period of restitution, $t_m < t < t_2$.

The principle of impulse and momentum is applied to A during the intervals of time from t_1 to the time of closest approach t_m and also from t_m to t_2,

$$\int_{t_1}^{t_m} -F_c \, dt = m_A v_m - m_A v_A \tag{3.52}$$

$$\int_{t_m}^{t_2} -F_r \, dt = m_A v'_A - m_A v_m, \tag{3.53}$$

where F_c is the magnitude of the contact force during the compression phase and F_r is the magnitude of the contact force during the restitution phase.

Then the principle of impulse and momentum is applied to B for the same intervals of time:

$$\int_{t_1}^{t_m} F_c + dt = m_B v_m - m_B v_B \tag{3.54}$$

$$\int_{t_m}^{t_2} F_r \, dt = m_B v'_B - m_B v_m, \tag{3.55}$$

As a result of the impact, part of the kinetic energy of the particles can be lost (because of a permanent deformation, generation of heat and sound, etc.). The impulse during the restitution phase of the impact from t_m to t_2 is in general smaller than the impulse during the compression phase t_1 to t_m.

The ratio of these impulses is called the *coefficient of restitution* (this definition was introduced by Poisson):

$$e = \frac{\int_{t_m}^{t_2} F_r \, dt}{\int_{t_1}^{t_m} F_c \, dt}. \tag{3.56}$$

The value of the coefficient of restitution depends on the properties of the objects as well as their velocities and orientations when they collide, and it can be determined by experiment or by a detailed analysis of the deformations of the objects during the impact.

If Eq. (3.53) is divided by Eq. (3.52) and Eq. (3.55) is divided by Eq. (3.54), the resulting equations are

$$(v_m - v_A)e = v'_A - v_m$$
$$(v_m - v_B)e = v'_B - v_m.$$

If the first equation is subtracted from the second one, the coefficient of restitution is

$$e = \frac{v'_B - v'_A}{v_A - v_B}. \tag{3.57}$$

Thus, the coefficient of restitution is related to the relative velocities of the objects before and after the impact (this is the kinematic definition of e introduced by Newton). If the coefficient of restitution e is known, Eq. (3.57) together with the equation of conservation of linear momentum, Eq. (3.51), may be used to determine v'_A and v'_B.

If $e = 0$ in Eq. (3.57), then $v'_B = v'_A$ and the objects remain together after the impact. The impact is perfectly plastic.

If $e = 1$, the total kinetic energy is the same before and after the impact:

$$\frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A (v'_A)^2 + \frac{1}{2} m_B (v'_B)^2.$$

An impact in which kinetic energy is conserved is called *perfectly elastic.*

3.13.2 OBLIQUE CENTRAL IMPACTS

Two small spheres A and B with the masses m_A and m_B approach with arbitrary velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ (Fig. 3.15a). The initial velocities are not parallel, but they are in the same plane.

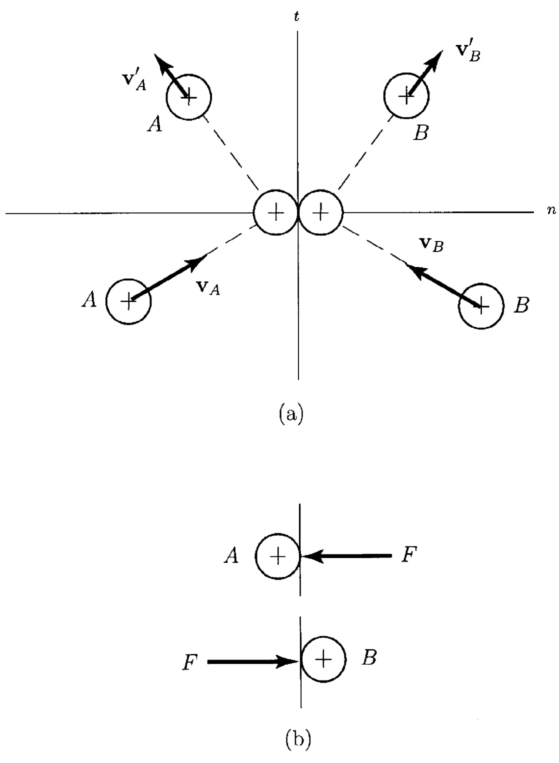

Figure 3.15

The forces they exert on each other during their impact are parallel to the n axis (center line axis) and point toward their centers of mass (Fig. 3.15b). There are no forces in the t direction at the contact point (tangent direction at the contact point). The velocities in the t direction are unchanged by the impact:

$$(\mathbf{v}'_A)_t = (\mathbf{v}_A)_t \quad \text{and} \quad (\mathbf{v}'_B)_t = (\mathbf{v}_B)_t. \tag{3.58}$$

In the n direction the linear momentum is conserved:

$$m_A(\mathbf{v}_A)_n + m_B(\mathbf{v}_B)_n = m_A(\mathbf{v}'_A)_n + m_B(\mathbf{v}'_B)_n. \tag{3.59}$$

The coefficient of restitution is defined as

$$e = \frac{(\mathbf{v}'_B)_n - (\mathbf{v}'_A)_n}{(\mathbf{v}_A)_n - (\mathbf{v}_B)_n}. \tag{3.60}$$

If B is a stationary object (fixed relative to the earth), then

$$(\mathbf{v}'_A)_n = -e(\mathbf{v}_A)_n.$$

3.14 Principle of Angular Impulse and Momentum

The position of a particle P of mass m relative to an inertial reference frame with origin O is given by the position vector $\mathbf{r} = \mathbf{OP}$ (Fig. 3.16). The cross product of Newton's second law with the position vector $\mathbf{r}$ is

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt}. \tag{3.61}$$

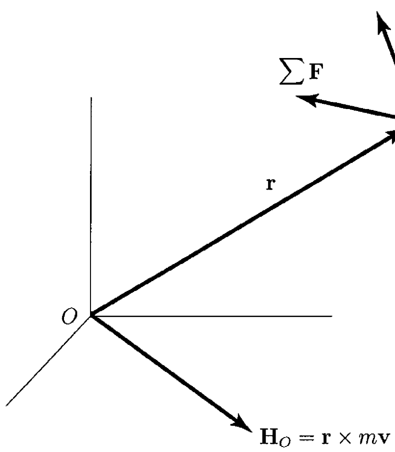

Figure 3.16

The time derivative of the quantity $\mathbf{r} \times m\mathbf{v}$ is

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) + \left(\mathbf{r} \times m\frac{d\mathbf{v}}{dt}\right) = \mathbf{r} \times m\frac{d\mathbf{v}}{dt},$$

because $d\mathbf{r}/dt = \mathbf{v}$, and the cross product of parallel vectors is zero. Equation (3.61) may be written as

$$\mathbf{r} \times \mathbf{F} = \frac{d\mathbf{H}_O}{dt}, \tag{3.62}$$

where the vector

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{3.63}$$

is called the *angular momentum* about O (Fig. 3.16). The angular momentum may be interpreted as the moment of the linear momentum of the particle about point O. The moment $\mathbf{r} \times \mathbf{F}$ equals the rate of change of the moment of momentum about point O.

Integrating Eq. (3.62) with respect to time, one may obtain

$$\int_{t_1}^{t_2} (\mathbf{r} \times \mathbf{F})\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1. \tag{3.64}$$

The integral on the left-hand side is called the *angular impulse*.

The principle of angular impulse and momentum: The angular impulse applied to a particle during an interval of time is equal to the change in its angular momentum.

The dimensions of the angular impulse and angular momentum are (mass) × (length)^2/(time).

4. Planar Kinematics of a Rigid Body

A rigid body is an idealized model of an object that does not deform, or change shape. A rigid body is by definition an object with the property that the distance between every pair of points of the rigid body is constant. Although any object does not deform as it moves, if its deformation is small one may approximate its motion by modeling it as a rigid body.

4.1 Types of Motion

The rigid body motion is described with respect to a reference frame (coordinate system) relative to which the motions of the points of the rigid body and its angular motion are measured. In many situations it is convenient to use a reference frame that is fixed with respect to the earth.

Rotation about a fixed axis. Each point of the rigid body on the axis is stationary, and each point not on the axis moves in a circular path about the axis as the rigid body rotates (Fig. 4.1a).

Translation. Each point of the rigid body describes parallel paths (Fig. 4.1b). Every point of a rigid body in translation has the same velocity and acceleration. The motion of the rigid body may be described the motion of a single point.

Planar motion. Consider a rigid body intersected by a plane fixed relative to a given reference frame (Fig. 4.1c). The points of the rigid body intersected by the plane remain in the plane for two-dimensional, or planar, motion. The fixed plane is the plane of the motion. Planar motion or complex motion exhibits a simultaneous combination of rotation and translation. Points on the rigid body will travel nonparallel paths, and there will be at every instant a center of rotation, which will continuously change location.

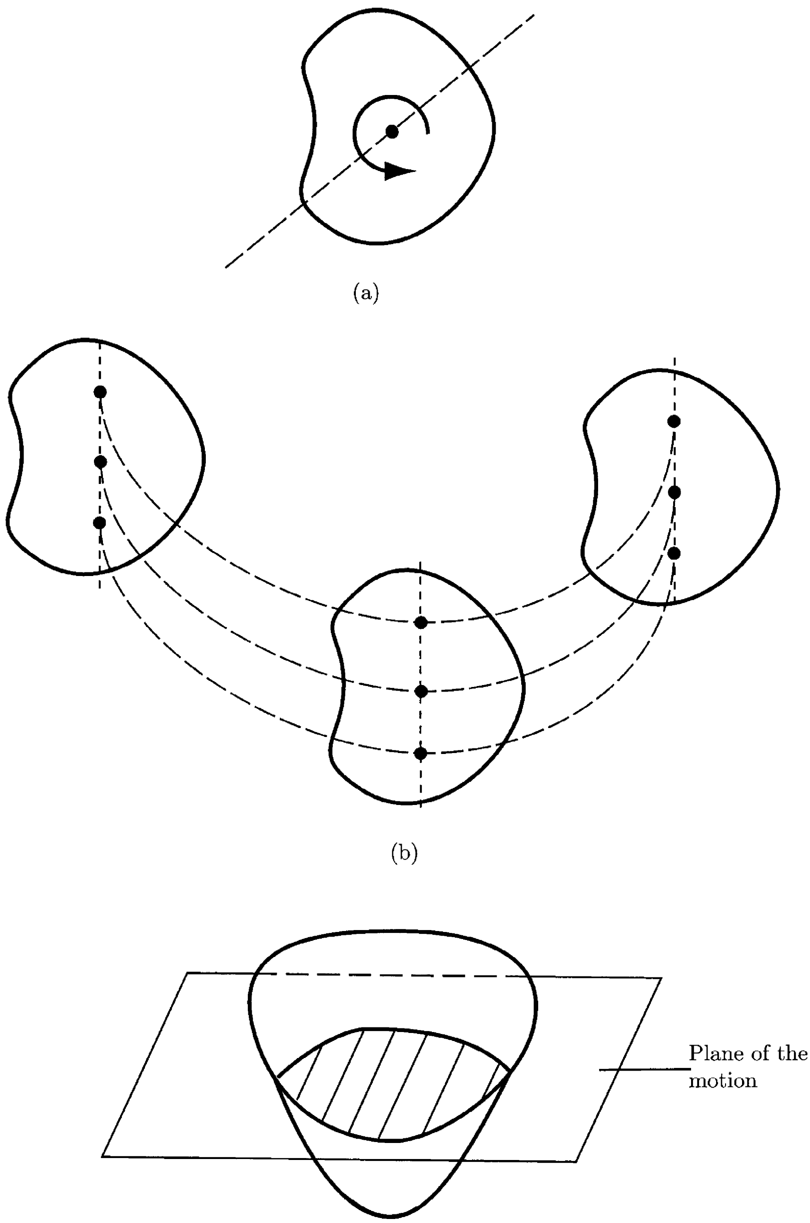

Figure 4.1

The rotation of a rigid body about a fixed axis is a special case of planar motion.

4.2 Rotation about a Fixed Axis

Figure 4.2 shows a rigid body rotating about a fixed axis a. The reference line b is fixed and is perpendicular to the fixed axis a, $b \perp a$. The body-fixed line

c rotates with the rigid body and is perpendicular to the fixed axis a, $c \perp a$. The angle θ between the reference line and the body-fixed line describes the position, or orientation, of the rigid body about the fixed axis. The angular velocity (rate of rotation) of the rigid body is

$$\omega = \frac{d\theta}{dt} = \dot{\theta}, \tag{4.1}$$

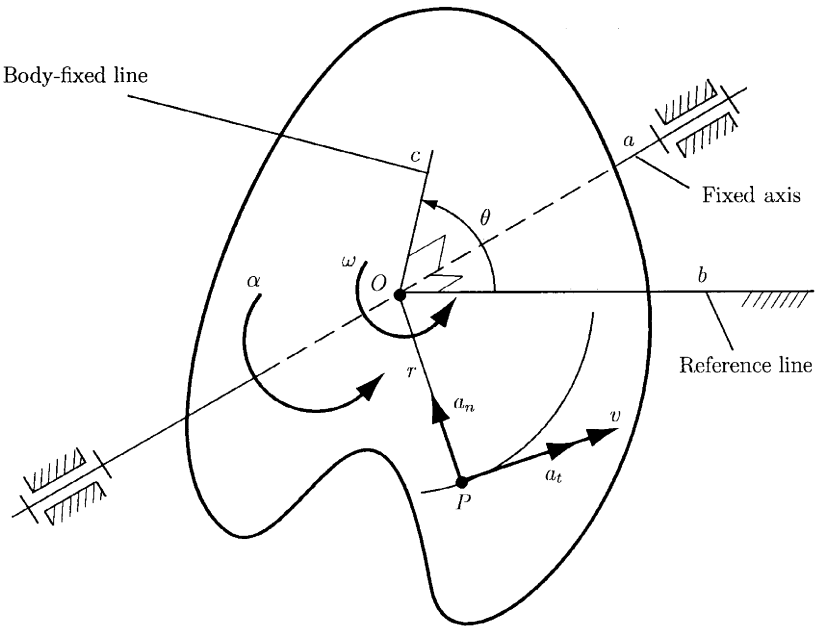

Figure 4.2

and the angular acceleration of the rigid body is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}. \tag{4.2}$$

The velocity of a point P, of the rigid body, at a distance r from the fixed axis is tangent to its circular path (Fig. 4.2) and is given by

$$v = r\omega. \tag{4.3}$$

The normal and tangential accelerations of P are

$$a_t = r\alpha, \quad a_n = \frac{v^2}{r} = r\omega^2. \tag{4.4}$$

4.3 Relative Velocity of Two Points of the Rigid Body

Figure 4.3 shows a rigid body in planar translation and rotation. The position vector of the point A of the rigid body $\mathbf{r}_A = \mathbf{OA}$, and the position vector of the point B of the rigid body is $\mathbf{r}_B = \mathbf{OB}$. The point O is the origin of a given reference frame. The position of point A relative to point B is the vector **BA**.

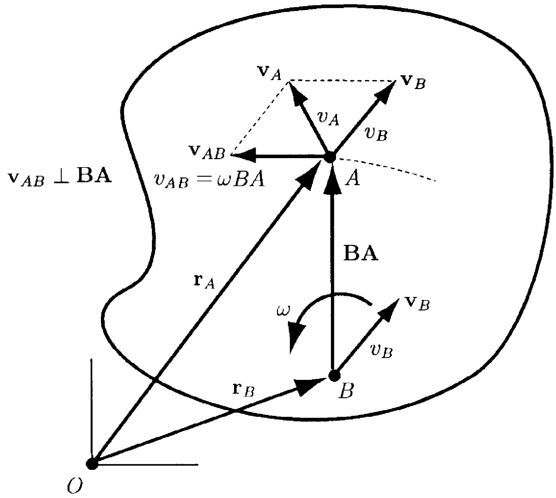

Figure 4.3

The position vector of point A relative to point B is related to the positions of A and B relative to O by

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{BA}. \tag{4.5}$$

The derivative of Eq. (4.5) with respect to time gives

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}, \tag{4.6}$$

where $\mathbf{v}_A$ and $\mathbf{v}_B$ are the velocities of A and B relative to the reference frame. The velocity of point A relative to point B is

$$\mathbf{v}_{AB} = \frac{d\mathbf{BA}}{dt}.$$

Since A and B are points of the rigid body, the distance between them, $BA = |\mathbf{BA}|$, is constant. That means that relative to B, A moves in a circular path as the rigid body rotates. The velocity of A relative to B is therefore tangent to the circular path and equal to the product of the angular velocity ω of the rigid body and BA:

$$v_{AB} = |\mathbf{v}_{AB}| = \omega BA. \tag{4.7}$$

The velocity $\mathbf{v}_{AB}$ is perpendicular to the position vector $\mathbf{BA}$, $\mathbf{v}_{AB} \perp \mathbf{BA}$. The sense of $\mathbf{v}_{AB}$ is the sense of ω (Fig. 4.3). The velocity of A is the sum of the velocity of B and the velocity of A relative to B.

4.4 Angular Velocity Vector of a Rigid Body

EULER'S THEOREM

A rigid body constrained to rotate about a fixed point can move between any two positions by a single rotation about some axis through the fixed point. ▲

With Euler's theorem the change in position of a rigid body relative to a fixed point A during an interval of time from t to $t + dt$ may be expressed as a single rotation through an angle $d\theta$ about some axis. At the time t the rate of rotation of the rigid body about the axis is its angular velocity $\omega = d\theta/dt$, and the axis about which it rotates is called the *instantaneous axis of rotation.*

The angular velocity vector of the rigid body, denoted by $\boldsymbol{\omega}$, specifies both the direction of the instantaneous axis of rotation and the angular velocity. The vector $\boldsymbol{\omega}$ is defined to be parallel to the instantaneous axis of rotation (Fig. 4.4), and its magnitude is the rate of rotation, the absolute value of ω. The direction of $\boldsymbol{\omega}$ is related to the direction of the rotation of the rigid body through a right-hand rule: if one points the thumb of the right hand in the direction of $\boldsymbol{\omega}$, the fingers curl around $\boldsymbol{\omega}$ in the direction of the rotation.

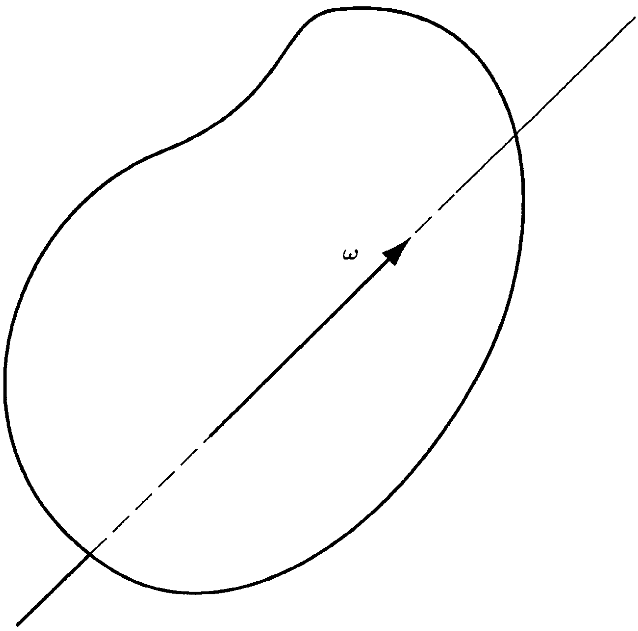

Figure 4.4

Figure 4.5 shows two points A and B of a rigid body. The rigid body has the angular velocity $\boldsymbol{\omega}$. The velocity of A relative to B is given by the equation

$$\mathbf{v}_{AB} = \frac{d\mathbf{BA}}{dt} = \boldsymbol{\omega} \times \mathbf{BA}. \tag{4.8}$$

Proof

The point A is moving at the present instant in a circular path relative to the point B. The radius of the path is $|\mathbf{BA}| \sin\beta$, where β is the angle between the vectors $\mathbf{BA}$ and $\boldsymbol{\omega}$. The magnitude of the velocity of A relative to B is equal to the product of the radius of the circular path and the angular velocity of the

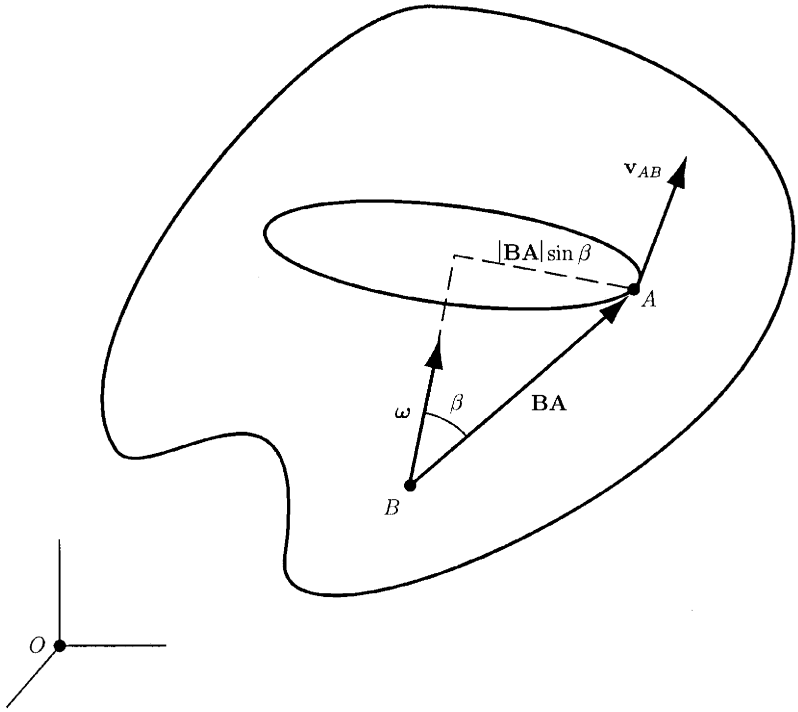

Figure 4.5

rigid body, $|\mathbf{v}_{AB}| = (|\mathbf{BA}|\sin\beta)|\omega|$, which is the magnitude of the cross product of **BA** and $\boldsymbol{\omega}$ or

$$\mathbf{v}_{AB} = \boldsymbol{\omega} \times \mathbf{BA}.$$

The relative velocity $\mathbf{v}_{AB}$ is perpendicular to $\boldsymbol{\omega}$ and perpendicular to **BA**.

When Eq. (4.8) is substituted into Eq. (4.6), the relation between the velocities of two points of a rigid body in terms of its angular velocity is obtained:

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB} = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{BA}. \tag{4.9}$$

▲

4.5 Instantaneous Center

The *instantaneous center* of a rigid body is a point whose velocity is zero at the instant under consideration. Every point of the rigid body rotates about the instantaneous center at the instant under consideration.

The instantaneous center may be or may not be a point of the rigid body. When the instantaneous center is not a point of the rigid body, the rigid body is rotating about an external point at that instant.

Figure 4.6 shows two points A and B of a rigid body and their directions of motion Δ_A amd Δ_B,

$$\mathbf{v}_A \| \Delta_A \quad \text{and} \quad \mathbf{v}_B \| \Delta_B,$$

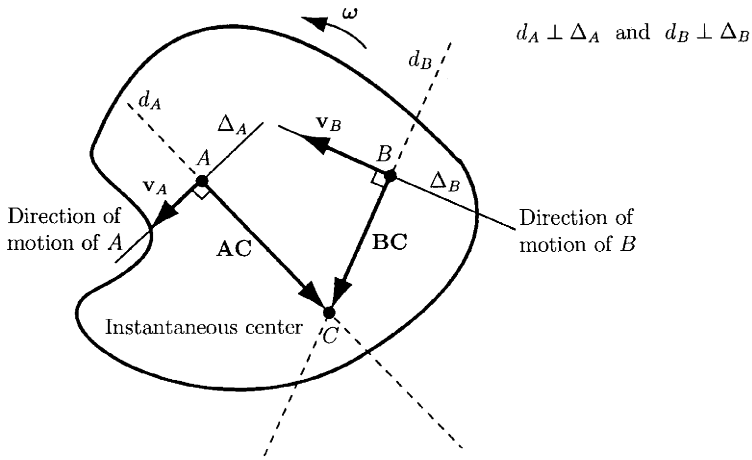

Figure 4.6

where $\mathbf{v}_A$ is the velocity of point A, and $\mathbf{v}_B$ is the velocity of point B.

Through the points A and B perpendicular lines are drawn to their directions of motion:

$$d_A \perp \Delta_A \quad \text{and} \quad d_B \perp \Delta_B.$$

The perpendicular lines intersect at point C:

$$d_A \cap d_B = C.$$

The velocity of point C in terms of the velocity of point A is

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{AC},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body. Since the vector $\boldsymbol{\omega} \times \mathbf{AC}$ is perpendicular to $\mathbf{AC}$,

$$(\boldsymbol{\omega} \times \mathbf{AC}) \perp \mathbf{AC},$$

this equation states that the direction of motion of C is parallel to the direction of motion of A:

$$\mathbf{v}_C \| \mathbf{v}_A. \tag{4.10}$$

The velocity of point C in terms of the velocity of point B is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{BC}.$$

The vector $\boldsymbol{\omega} \times \mathbf{BC}$ is perpendicular to $\mathbf{BC}$,

$$(\boldsymbol{\omega} \times \mathbf{BC}) \perp \mathbf{BC},$$

so this equation states that the direction of motion of C is parallel to the direction of motion of B:

$$\mathbf{v}_C \| \mathbf{v}_B. \tag{4.11}$$

But C cannot be moving parallel to A and parallel to B, so Eqs. (4.10) and (4.11) are contradictory unless $\mathbf{v}_C = \mathbf{0}$. So the point C, where the perpendicular lines through A and B to their directions of motion intersect, is the instantaneous center. This is a simple method to locate the instantaneous center of a rigid body in planar motion.

If the rigid body is in translation (the angular velocity of the rigid body is zero) the instantaneous center of the rigid body C moves to infinity.

4.6 Relative Acceleration of Two Points of the Rigid Body

The velocities of two points A and B of a rigid body in planar motion relative to a given reference frame with the origin at point O are related by (Fig. 4.7)

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}.$$

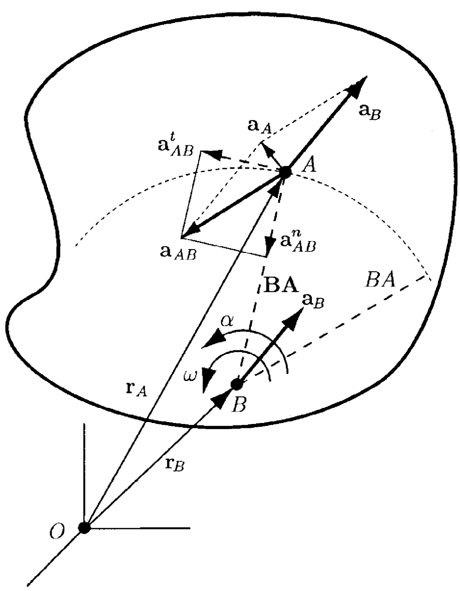

Figure 4.7

Taking the time derivative of this equation, one may obtain

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{AB},$$

where $\mathbf{a}_A$ and $\mathbf{a}_B$ are the accelerations of A and B relative to the origin O of the reference frame and $\mathbf{a}_{AB}$ is the acceleration of point A relative to point B. Because the point A moves in a circular path relative to the point B as the rigid body rotates, $\mathbf{a}_{AB}$ has a normal component and a tangential component (Fig. 4.7):

$$\mathbf{a}_{AB} = \mathbf{a}^n_{AB} + \mathbf{a}^t_{AB}.$$

The normal component points toward the center of the circular path (point B), and its magnitude is

$$|\mathbf{a}^n_{AB}| = |\mathbf{v}_{AB}|^2/|\mathbf{BA}| = \omega^2 BA.$$

The tangential component equals the product of the distance $BA = |\mathbf{BA}|$ and the angular acceleration α of the rigid body:

$$|\mathbf{a}^t_{AB}| = \alpha BA.$$

The velocity of the point A relative to the point B in terms of the angular velocity vector, $\boldsymbol{\omega}$, of the rigid body is given by Eq. (4.8):

$$\mathbf{v}_{AB} = \boldsymbol{\omega} \times \mathbf{BA}.$$

Taking the time derivative of this equation, one may obtain

$$\begin{aligned}\mathbf{a}_{AB} &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{BA} + \boldsymbol{\omega} \times \mathbf{v}_{AB} \\ &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA}).\end{aligned}$$

Defining the angular acceleration vector $\boldsymbol{\alpha}$ to be the rate of change of the angular velocity vector,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}, \tag{4.12}$$

one finds that the acceleration of A relative to B is

$$\mathbf{a}_{AB} = \boldsymbol{\alpha} \times \mathbf{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA}).$$

The velocities and accelerations of two points of a rigid body in terms of its angular velocity and angular acceleration are

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{BA} \tag{4.13}$$

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA}). \tag{4.14}$$

In the case of planar motion, the term $\boldsymbol{\alpha} \times \mathbf{BA}$ in Eq. (4.14) is the tangential component of the acceleration of A relative to B, and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA})$ is the normal component (Fig. 4.7). Equation (4.14) may be written for planar motion in the form

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{BA} - \omega^2 \mathbf{BA}. \tag{4.15}$$

4.7 Motion of a Point That Moves Relative to a Rigid Body

A reference frame that moves with the rigid body is a *body fixed* reference frame. Figure 4.8 shows a rigid body RB, in motion relative to a primary reference frame with its origin at point O_0, XO_0YZ. The primary reference frame is a fixed reference frame or an earth fixed reference frame. The unit vectors $\boldsymbol{\imath}_0$, $\boldsymbol{\jmath}_0$, and $\mathbf{k}_0$ of the primary reference frame are constant.

The body fixed reference frame, $xOyz$, has its origin at a point O of the rigid body ($O \in RB$) and is a moving reference frame relative to the primary

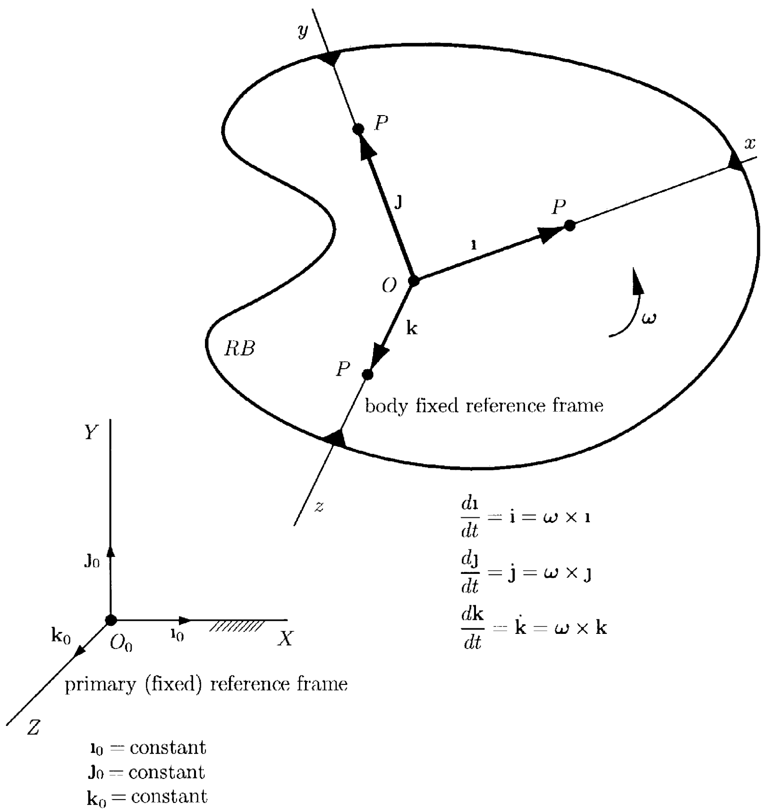

Figure 4.8

reference. The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame.

The position vector of a point P of the rigid body ($P \in RB$) relative to the origin, O, of the body fixed reference frame is the vector $\mathbf{OP}$. The velocity of P relative to O is

$$\frac{d\mathbf{OP}}{dt} = \mathbf{v}_{PO} = \boldsymbol{\omega} \times \mathbf{OP},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body. The unit vector $\mathbf{i}$ may be regarded as the position vector of a point P of the rigid body (Fig. 4.8), and its time derivative may be written as $d\mathbf{i}/dt = \dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i}$. In a similar way the time derivative of the unit vectors $\mathbf{j}$ and $\mathbf{k}$ may be obtained. The expressions

$$\begin{aligned} \frac{d\mathbf{i}}{dt} &= \dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i} \\ \frac{d\mathbf{j}}{dt} &= \dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j} \\ \frac{d\mathbf{k}}{dt} &= \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k} \end{aligned} \tag{4.16}$$

are known as Poisson's relations.

The position vector of a point A (the point A is not assumed to be a point of the rigid body) relative to the origin O_0 of the primary reference frame is (Fig. 4.9)

$$\mathbf{r}_A = \mathbf{r}_O + \mathbf{r},$$

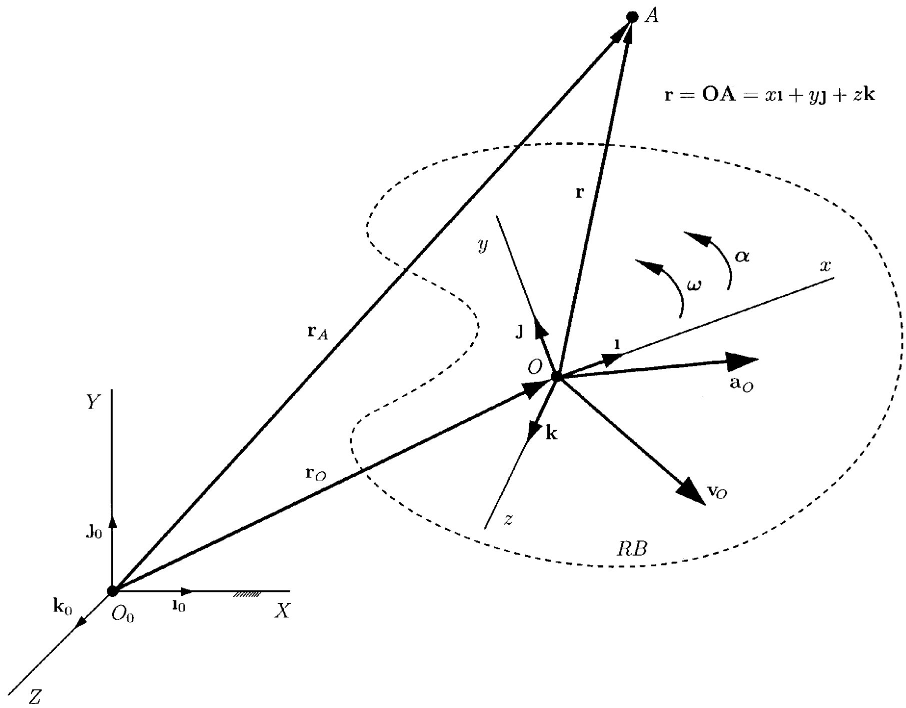

Figure 4.9

where

$$\mathbf{r} = \mathbf{OA} = x\boldsymbol{\imath} + y\boldsymbol{\jmath} + z\mathbf{k}$$

is the position vector of A relative to the origin O, of the body fixed reference frame, and x, y, and z are the coordinates of A in terms of the body fixed reference frame. The velocity of the point A is the time derivative of the position vector $\mathbf{r}_A$:

$$\mathbf{v}_A = \frac{d\mathbf{r}_O}{dt} + \frac{d\mathbf{r}}{dt} = \mathbf{v}_O + \mathbf{v}_{AO}$$
$$= \mathbf{v}_O + \frac{dx}{dt}\boldsymbol{\imath} + x\frac{d\boldsymbol{\imath}}{dt} + \frac{dy}{dt}\boldsymbol{\jmath} + y\frac{d\boldsymbol{\jmath}}{dt} + \frac{dz}{dt}\mathbf{k} + z\frac{d\mathbf{k}}{dt}.$$

Using Eqs. (4.16), one finds that the total derivative of the position vector $\mathbf{r}$ is

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k} + \boldsymbol{\omega} \times \mathbf{r}.$$

The velocity of A relative to the body fixed reference frame is a local derivative:

$$\mathbf{v}_{Arel} = \frac{\partial \mathbf{r}}{\partial t} = \frac{dx}{dt}\mathbf{\imath} + \frac{dy}{dt}\mathbf{\jmath} + \frac{dz}{dt}\mathbf{k} = \dot{x}\mathbf{\imath} + \dot{y}\mathbf{\jmath} + \dot{z}\mathbf{k}, \tag{4.17}$$

A general formula for the total derivative of a moving vector $\mathbf{r}$ may be written as

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial t} + \boldsymbol{\omega} \times \mathbf{r}. \tag{4.18}$$

This relation is known as the *transport theorem*. In operator notation the transport theorem is written as

$$\frac{d}{dt}(\ ) = \frac{\partial}{\partial t}(\ ) + \boldsymbol{\omega} \times (\ ). \tag{4.19}$$

The velocity of the point A relative to the primary reference frame is

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}, \tag{4.20}$$

Equation (4.20) expresses the velocity of a point A as the sum of three terms:

- The velocity of a point O of the rigid body
- The velocity $\mathbf{v}_{Arel}$ of A relative to the rigid body
- The velocity $\boldsymbol{\omega} \times \mathbf{r}$ of A relative to O due to the rotation of the rigid body

The acceleration of the point A relative to the primary reference frame is obtained by taking the time derivative of Eq. (4.20):

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO} \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \end{aligned} \tag{4.21}$$

where

$$\mathbf{a}_{Arel} = \frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{d^2 x}{dt^2}\mathbf{\imath} + \frac{d^2 y}{dt^2}\mathbf{\jmath} + \frac{d^2 z}{dt^2}\mathbf{k} \tag{4.22}$$

is the acceleration of A relative to the body fixed reference frame or relative to the rigid body. The term

$$\mathbf{a}_{Cor} = 2\boldsymbol{\omega} \times \mathbf{v}_{Arel}$$

is called the Coriolis acceleration force.

In the case of planar motion, Eq. (4.21) becomes

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO} \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} - \boldsymbol{\omega}^2 \mathbf{r}, \end{aligned} \tag{4.23}$$

The motion of the rigid body (RB) is described relative to the primary reference frame. The velocity $\mathbf{v}_A$ and the acceleration $\mathbf{a}_A$ of point A are relative to the primary reference frame. The terms $\mathbf{v}_{Arel}$ and $\mathbf{a}_{Arel}$ are the velocity and acceleration of point A relative to the body fixed reference

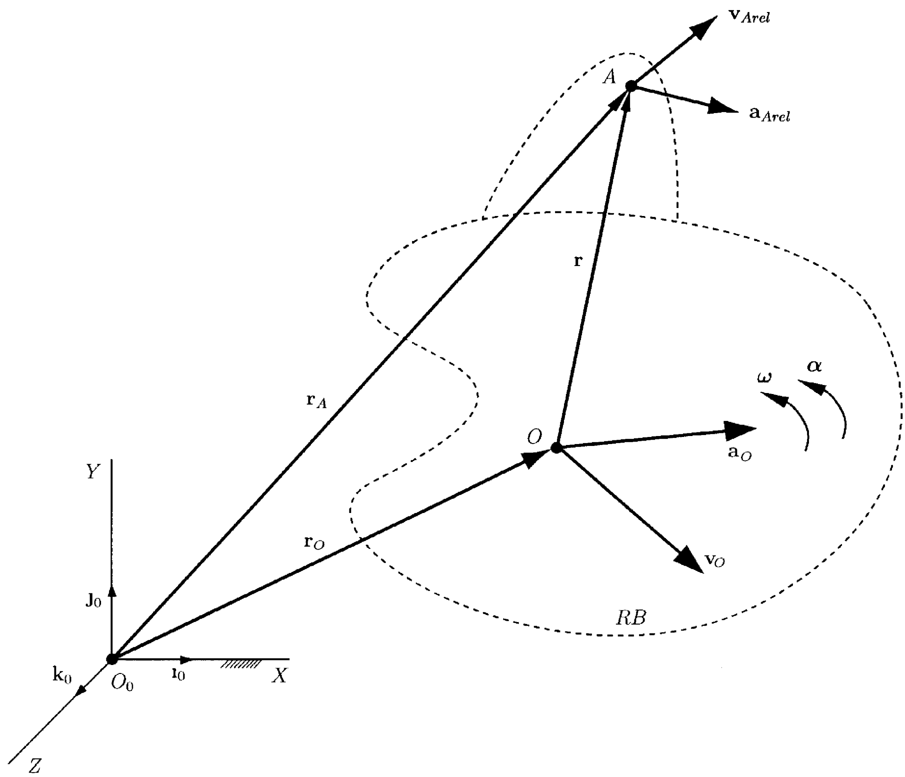

Figure 4.10

frame, i.e., they are the velocity and acceleration measured by an observer moving with the rigid body (Fig. 4.10).

If A is a point of the rigid body, $A \in RB$, $\mathbf{v}_{Arel} = \mathbf{0}$ and $\mathbf{a}_{Arel} = \mathbf{0}$.

4.7.1 MOTION OF A POINT RELATIVE TO A MOVING REFERENCE FRAME

The velocity and acceleration of an arbitrary point A relative to a point O of a rigid body, in terms of the body fixed reference frame, are given by Eqs. (4.20) and (4.21):

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{OA} \tag{4.24}$$

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{OA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OA}). \tag{4.25}$$

These results apply to any reference frame having a moving origin O and rotating with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ relative to a primary reference frame (Fig. 4.11). The terms $\mathbf{v}_A$ and $\mathbf{a}_A$ are the velocity and acceleration of an arbitrary point A relative to the primary reference frame. The terms $\mathbf{v}_{Arel}$ and $\mathbf{a}_{Arel}$ are the velocity and acceleration of A relative to the secondary moving reference frame, i.e., they are the velocity and acceleration measured by an observer moving with the secondary reference frame.

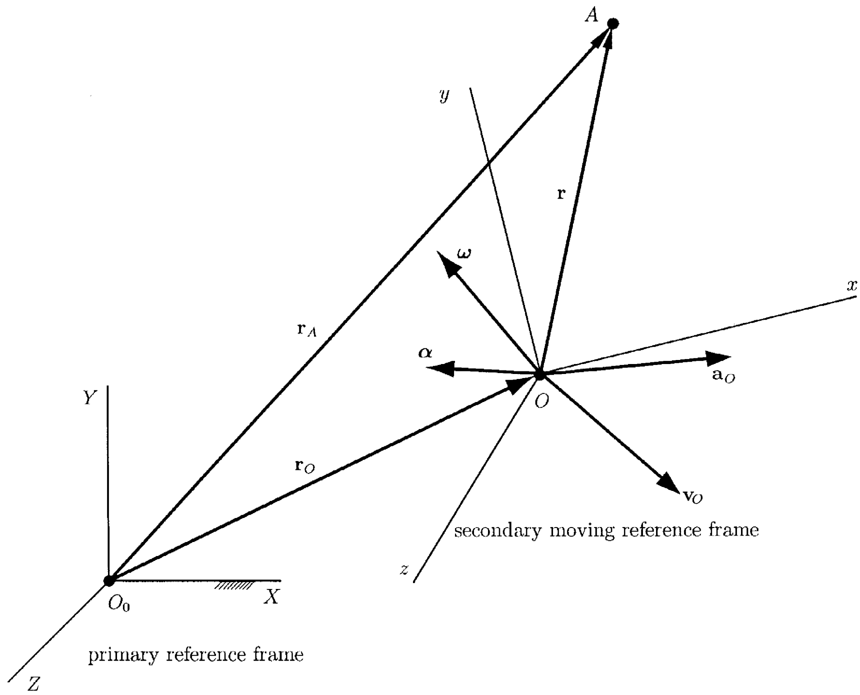

Figure 4.11

4.7.2 INERTIAL REFERENCE FRAMES

A reference frame is inertial if one may use it to apply Newton's second law in the form $\sum \mathbf{F} = m\mathbf{a}$.

Figure 4.12 shows a nonaccelerating, nonrotating reference frame with the origin at O_0, and a secondary nonrotating, earth centered reference frame with the origin at O. The nonaccelerating, nonrotating reference frame with the origin at O_0 is assumed to be an inertial reference. The acceleration of the earth, due to the gravitational attractions of the sun, moon, etc., is $\mathbf{g}_O$. The earth centered reference frame has the acceleration $\mathbf{g}_O$ as well.

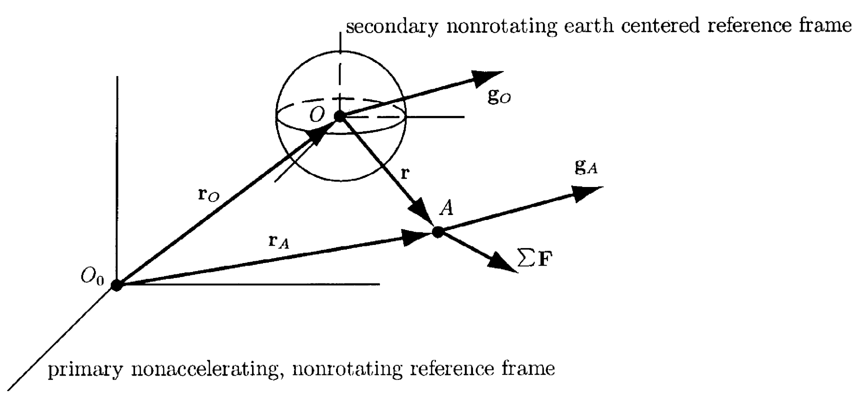

Figure 4.12

Newton's second law for an object A of mass m, using the hypothetical nonaccelerating, nonrotating reference frame with the origin at O_0, may be written as

$$m\mathbf{a}_A = m\mathbf{g}_A + \sum \mathbf{F}, \tag{4.26}$$

where $\mathbf{a}_A$ is the acceleration of A relative to O_0, $\mathbf{g}_A$ is the resulting gravitational acceleration, and $\sum \mathbf{F}$ is the sum of all other external forces acting on A.

By Eq. (4.25) the acceleration of A relative to O_0 is

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel},$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the earth centered reference frame and the acceleration of the origin O is equal to the gravitational acceleration of the earth, $\mathbf{a}_O = \mathbf{g}_O$. The earth-centered reference frame does not rotate ($\boldsymbol{\omega} = \mathbf{0}$).

If the object A is on or near the earth, its gravitational acceleration $\mathbf{g}_A$ due to the attraction of the sun, etc., is nearly equal to the gravitational acceleration of the earth $\mathbf{g}_O$, and Eq. (4.26) becomes

$$\sum \mathbf{F} = m\mathbf{a}_{Arel}. \tag{4.27}$$

One may apply Newton's second law using a nonrotating, earth centered reference frame if the object is near the earth.

In most applications, Newton's second law may be applied using an earth fixed reference frame. Figure 4.13 shows a nonrotating reference frame with its origin at the center of the earth O and a secondary earth fixed reference frame with its origin at a point B. The earth fixed reference frame with the origin at B may be assumed to be an inertial reference and

$$\sum \mathbf{F} = m\mathbf{a}_{Arel}, \tag{4.28}$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the earth fixed reference frame.

The motion of an object A may be analysed using a primary inertial reference frame with its origin at the point O (Fig. 4.14). A secondary reference frame with its origin at B undergoes an arbitrary motion with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. Newton's second law for the object A of mass m is

$$\sum \mathbf{F} = m\mathbf{a}_A, \tag{4.29}$$

where $\mathbf{a}_A$ is the acceleration of A acceleration relative to O. Equation (4.29) may be written in the form

$$\begin{aligned}\sum \mathbf{F} - m[\mathbf{a}_B + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{BA} \\ + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA})] = m\mathbf{a}_{Arel},\end{aligned} \tag{4.30}$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the secondary reference frame. The term $\mathbf{a}_B$ is the acceleration of the origin B of the secondary reference frame relative to the primary inertial reference. The term $2\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is the Coriolis acceleration, and the term $-2m\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is called the Coriolis force.

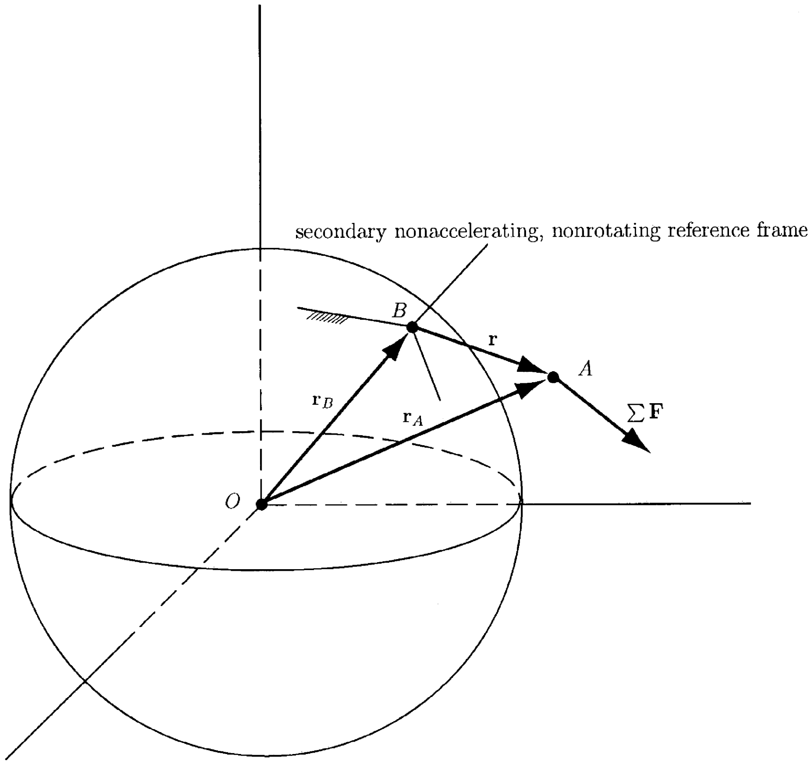

Figure 4.13 primary nonrotating earth centered reference

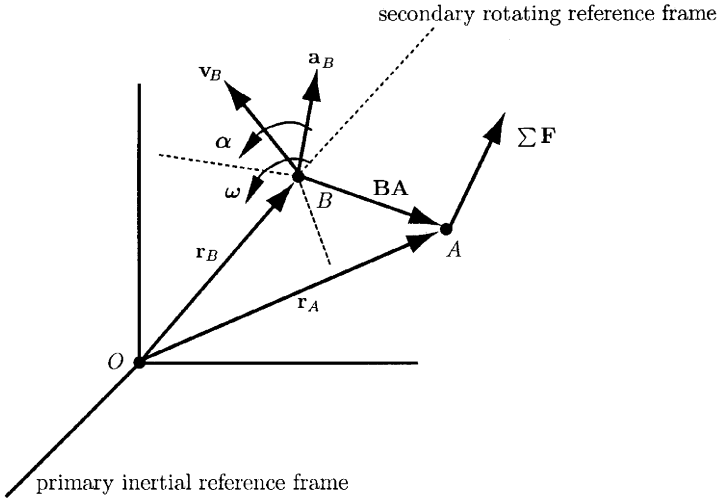

Figure 4.14 primary inertial reference frame

This is Newton's second law expressed in terms of a secondary reference frame undergoing an arbitrary motion relative to an inertial primary reference frame.

5. Dynamics of a Rigid Body

5.1 Equation of Motion for the Center of Mass

Newton stated that the total force on a particle is equal to the rate of change of its linear momentum, which is the product of its mass and velocity. Newton's second law is postulated for a particle, or small element of matter. One may show that the total external force on an arbitrary rigid body is equal to the product of its mass and the acceleration of its center of mass. An arbitrary rigid body with the mass m may be divided into N particles. The position vector of the i particle is $\mathbf{r}_i$ and the mass of the i particle is m_i (Fig. 5.1):

$$m = \sum_{i=1}^{N} m_i.$$

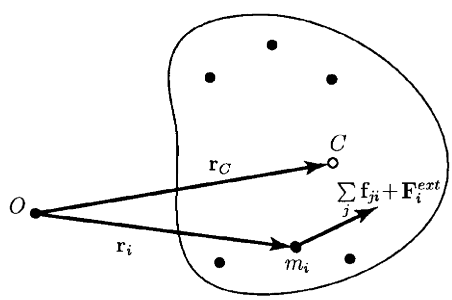

Figure 5.1

The position of the center of mass of the rigid body is

$$\mathbf{r}_C = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{m}. \tag{5.1}$$

Taking two time derivatives of Eq. (5.1), one may obtain

$$\sum_{i=1}^{N} m_i \frac{d^2\mathbf{r}_i}{dt^2} = m\frac{d^2\mathbf{r}_C}{dt^2} = m\mathbf{a}_C, \tag{5.2}$$

where $\mathbf{a}_C$ is the acceleration of the center of mass of the rigid body.

Let $\mathbf{f}_{ij}$ be the force exerted on the j particle by the i particle. Newton's third law states that the j particle exerts a force on the i particle of equal magnitude and opposite direction (Fig. 5.1):

$$\mathbf{f}_{ji} = -\mathbf{f}_{ij}.$$

Newton's second law for the i particle is

$$\sum_j \mathbf{f}_{ji} + \mathbf{F}_i^{ext} = m_i \frac{d^2\mathbf{r}_i}{dt^2}, \tag{5.3}$$

where $\mathbf{F}_i^{ext}$ is the external force on the i particle. Equation (5.3) may be written for each particle of the rigid body. Summing the resulting equations from $i = 1$ to N, one may obtain

$$\sum_i \sum_j \mathbf{f}_{ji} + \sum_i \mathbf{F}_i^{ext} = m\mathbf{a}_C, \tag{5.4}$$

The sum of the internal forces on the rigid body is zero (Newton's third law):

$$\sum_i \sum_j \mathbf{f}_{ji} = \mathbf{0}.$$

The term $\sum_i \mathbf{F}_i^{ext}$ is the sum of the external forces on the rigid body:

$$\sum_i \mathbf{F}_i^{ext} = \sum \mathbf{F}.$$

One may conclude that the sum of the external forces equals the product of the mass and the acceleration of the center of mass:

$$\sum \mathbf{F} = m\mathbf{a}_C. \tag{5.5}$$

If the rigid body rotates about a fixed axis O (Fig. 5.2), the sum of the moments about the axis due to external forces and couples acting on the body is

$$\sum M_O = I_O\alpha,$$

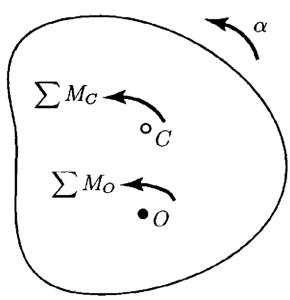

Figure 5.2

where I_O is the moment of inertia of the rigid body about O and α is the angular acceleration of the rigid body. In the case of general planar motion, the sum of the moments about the center of mass of a rigid body is related to its angular acceleration by

$$\sum M_C = I_C\alpha, \tag{5.6}$$

where I_C is the moment of inertia of the rigid body about its center of mass C.

If the external forces and couples acting on a rigid body in planar motion are known, one may use Eqs. (5.5) and (5.6) to determine the acceleration of the center of mass of the rigid body and the angular acceleration of the rigid body.

5.2 Angular Momentum Principle for a System of Particles

An arbitrary system with mass m may be divided into N particles $P_1, P_2, \ldots, P_N$. The position vector of the i particle is $\mathbf{r}_i = \mathbf{OP}_i$ and the mass of the i particle is m_i (Fig. 5.3). The position of the center of mass, C, of the system is $\mathbf{r}_C = \sum_{i=1}^{N} m_i \mathbf{r}_i / m$. The position of the particle P_i of the system relative to O is

$$\mathbf{r}_i = \mathbf{r}_C + \mathbf{CP}_i. \tag{5.7}$$

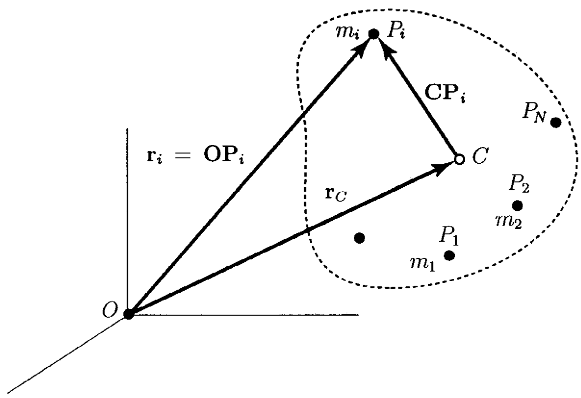

Figure 5.3

Multiplying Eq. (5.7) by m_i, summing from 1 to N, one may find that

$$\sum_{i=1}^{N} m_i \mathbf{CP}_i = \mathbf{0}. \tag{5.8}$$

The total angular momentum of the system about its center of mass C is the sum of the angular momenta of the particles about C,

$$\mathbf{H}_C = \sum_{i=1}^{N} \mathbf{CP}_i \times m_i \mathbf{v}_i, \tag{5.9}$$

where $\mathbf{v}_i = d\mathbf{r}_i/dt$ is the velocity of the particle P_i.

The total angular momentum of the system about O is the sum of the angular momenta of the particles,

$$\mathbf{H}_O = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_{i=1}^{N} (\mathbf{r}_C + \mathbf{CP}_i) \times m_i \mathbf{v}_i = \mathbf{r}_C \times m \mathbf{v}_C + \mathbf{H}_C, \tag{5.10}$$

or the total angular momentum about O is the sum of the angular momentum about O due to the velocity $\mathbf{v}_C$ of the center of mass of the system and the total angular momentum about the center of mass (Fig. 5.4).

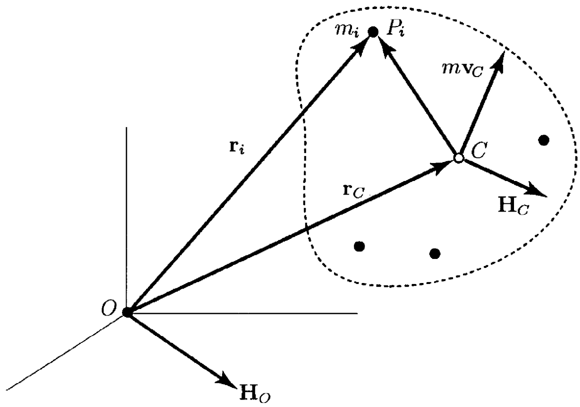

Figure 5.4

Newton's second law for the i particle is

$$\sum_j \mathbf{f}_{ji} + \mathbf{F}_i^{ext} = m_i \frac{d\mathbf{v}_i}{dt},$$

and the cross product with the position vector $\mathbf{r}_i$, and sum from $i = 1$ to N gives

$$\sum_i \sum_j \mathbf{r}_i \times \mathbf{f}_{ji} + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \sum_i \mathbf{r}_i \times \frac{d}{dt}(m_i \mathbf{v}_i). \tag{5.11}$$

The first term on the left side of Eq. (5.11) is the sum of the moments about O due to internal forces, and

$$\mathbf{r}_i \times \mathbf{f}_{ji} + \mathbf{r}_i \times \mathbf{f}_{ij} = \mathbf{r}_i \times (\mathbf{f}_{ji} + \mathbf{f}_{ij}) = \mathbf{0}.$$

The term vanishes if the internal forces between each pair of particles are equal, opposite, and directed along the straight line between the two particles (Fig. 5.5).

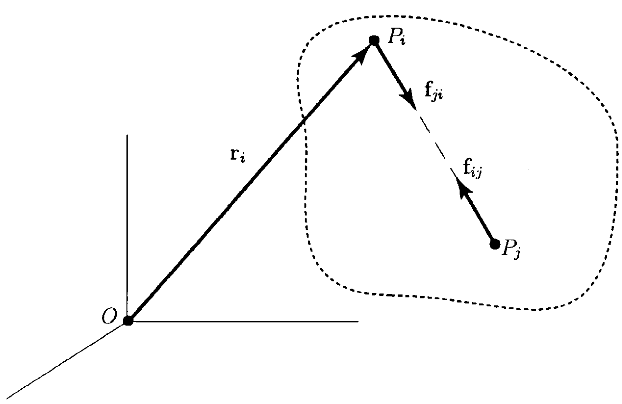

Figure 5.5

The second term on the left side of Eq. (5.11),

$$\sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \sum \mathbf{M}_O,$$

represents the sum of the moments about O due to the external forces and couples. The term on the right side of Eq. (5.11) is

$$\sum_i \mathbf{r}_i \times \frac{d}{dt}(m_i\mathbf{v}_i) = \sum_i \left[\frac{d}{dt}(\mathbf{r}_i \times m_i\mathbf{v}_i) - \mathbf{v}_i \times m_i\mathbf{v}_i\right] = \frac{d\mathbf{H}_O}{dt}, \qquad (5.12)$$

which represents the rate of change of the total angular momentum of the system about the point O.

Equation (5.11) may be rewritten as

$$\sum \mathbf{M}_O = \frac{d\mathbf{H}_O}{dt}. \qquad (5.13)$$

The sum of the moments about O due to external forces and couples equals the rate of change of the angular momentum about O.

Using Eqs. (5.10) and (5.13), one may obtain

$$\sum \mathbf{M}_O = \frac{d}{dt}(\mathbf{r}_C \times m\mathbf{v}_C + \mathbf{H}_C) = \mathbf{r}_C \times m\mathbf{a}_C + \frac{d\mathbf{H}_C}{dt}, \qquad (5.14)$$

where $\mathbf{a}_C$ is the acceleration of the center of mass.

If the point O is coincident with the center of mass at the present instant $C = O$, then $\mathbf{r}_C = \mathbf{0}$ and Eq. (5.14) becomes

$$\sum \mathbf{M}_C = \frac{d\mathbf{H}_C}{dt}. \qquad (5.15)$$

The sum of the moments about the center of mass equals the rate of change of the angular momentum about the center of mass.

5.3 Equations of Motion for General Planar Motion

An arbitrary rigid body with the mass m may be divided into N particles P_i, $i = 1, 2, \ldots, N$. The position vector of the P_i particle is $\mathbf{r}_i = \mathbf{OP}_i$ and the mass of the particle is m_i (Fig. 5.6).

Let d_O be the axis through the fixed origin point O that is perpendicular to the plane of the motion of a rigid body x, y, $d_O \perp (x, y)$. Let d_C be the parallel axis through the center of mass C, $d_C \| d_O$. The rigid body has a general planar motion, and one may express the angular velocity vector as $\boldsymbol{\omega} = \omega\mathbf{k}$.

The velocity of the P_i particle relative to the center of mass is

$$\frac{d\mathbf{R}_i}{dt} = \omega\mathbf{k} \times \mathbf{R}_i,$$

where $\mathbf{R}_i = \mathbf{CP}_i$. The sum of the moments about O due to external forces and couples is

$$\sum \mathbf{M}_O = \frac{d\mathbf{H}_O}{dt} = \frac{d}{dt}[(\mathbf{r}_C \times m\mathbf{v}_C) + \mathbf{H}_C], \qquad (5.16)$$

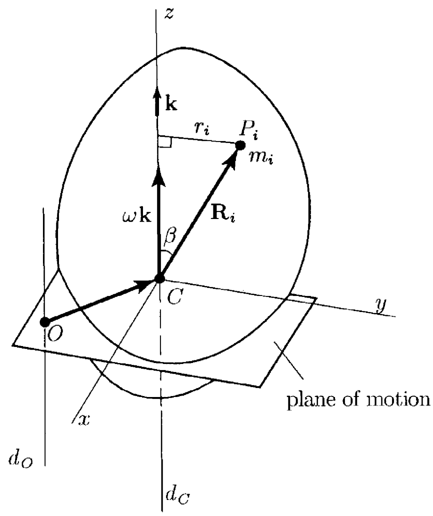

Figure 5.6

where

$$\mathbf{H}_C = \sum_i [\mathbf{R}_i \times m_i(\omega\mathbf{k} \times \mathbf{R}_i)]$$

is the angular momentum about d_C. The magnitude of the angular momentum about d_C is

$$\begin{aligned} H_C = \mathbf{H}_C \cdot \mathbf{k} &= \sum_i [\mathbf{R}_i \times m_i(\omega\mathbf{k} \times \mathbf{R}_i)] \cdot \mathbf{k} \\ &= \sum_i m_i[(\mathbf{R}_i \times \mathbf{k}) \times \mathbf{R}_i)] \cdot \mathbf{k}\omega = \sum_i m_i[(\mathbf{R}_i \times \mathbf{k}) \cdot (\mathbf{R}_i \times \mathbf{k})]\omega \\ &= \sum_i m_i|\mathbf{R}_i \times \mathbf{k}|^2\omega = \sum_i m_i r_i \omega, \end{aligned} \tag{5.17}$$

where the term $|\mathbf{k} \times \mathbf{R}_i| = r_i$ is the perpendicular distance from d_C to the P_i particle. The identity

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

has been used.

The moment of inertia of the rigid body about d_C is

$$I = \sum_i m_i r_i^2,$$

Equation (5.17) defines the angular momentum of the rigid body about d_C:

$$H_C = I\omega \quad \text{or} \quad \mathbf{H}_C = I\omega\mathbf{k} = I\boldsymbol{\omega}.$$

Substituting this expression into Eq. (5.16), one may obtain

$$\sum \mathbf{M}_O = \frac{d}{dt}[(\mathbf{r}_C \times m\mathbf{v}_C) + I\boldsymbol{\omega}] = (\mathbf{r}_C \times m\mathbf{a}_C) + I\boldsymbol{\alpha}. \tag{5.18}$$

If the fixed axis d_O is coincident with d_C at the present instant, $\mathbf{r} = \mathbf{0}$, and from Eq. (5.18) one may obtain

$$\sum \mathbf{M}_C = I\boldsymbol{\alpha}.$$

The sum of the moments about d_C equals the product of the moment of inertia about d_C and the angular acceleration.

5.4 D'Alembert's Principle

Newton's second law may be written as

$$\mathbf{F} + (-m\mathbf{a}_C) = \mathbf{0}, \quad \text{or} \quad \mathbf{F} + \mathbf{F}_{in} = \mathbf{0}, \tag{5.19}$$

where the term $\mathbf{F}_{in} = -m\mathbf{a}_C$ is the *inertial force*. Newton's second law may be regarded as an "equilibrium" equation.

Equation (5.18) relates the total moment about a fixed point O to the acceleration of the center of mass and the angular acceleration:

$$\sum \mathbf{M}_O = (\mathbf{r}_C \times m\mathbf{a}_C) + I\boldsymbol{\alpha}$$

or

$$\sum \mathbf{M}_O + [\mathbf{r}_C \times (-m\mathbf{a}_C)] + (-I\boldsymbol{\alpha}) = \mathbf{0}. \tag{5.20}$$

The term $\mathbf{M}_{in} = -I\boldsymbol{\alpha}$ is the *inertial couple*. The sum of the moments about any point, including the moment due to the inertial force $-m\mathbf{a}$ acting at the center of mass and the inertial couple, equals zero.

The equations of motion for a rigid body are analogous to the equations for static equilibrium: The sum of the forces equals zero and the sum of the moments about any point equals zero when the inertial forces and couples are taking into account. This is called *D'Alembert's principle*.

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