Chapter 9
Applications of Partial Differential Equations in Mechanical Engineering Analysis

Chapter Learning Objectives

- Learn the physical meaning of partial derivatives of functions.
- Learn that there are different order of partial derivatives describing the rate of changes of functions representing real physical quantities.
- Learn the two commonly used techniques for solving partial differential equations by (1) Integral transform methods that include the Laplace transform for physical problems covering half-space, and the Fourier transform method for problems that cover the entire space; (2) the separation of variables technique.
- Learn the use of the separation of variables technique to solve partial differential equations relating to heat conduction in solids and vibration of solids in multidimensional systems.

9.1 Introduction

Partial differential equations such as that shown in Equation (2.5) are the equations that involve partial derivatives described in Section 2.2.5. A partial derivative represents the rate of change of a function (a physical quantity in engineering analysis) with respect to one of several variables that the function is associated with.

The independent variables in partial derivatives can be (1) *spatial* variables represented by (x,y,z) in a rectangular coordinate system or (r,θ,z) in a cylindrical polar coordinate system and (2) *temporal* variables represented by time t.

Partial differential equations can be categorized as "boundary-value problems" or "initial-value problems", or "initial-boundary value problems". *Boundary-value problems* are the ones for which the complete solution of the partial differential equation is possible with specific boundary conditions. *Initial-value problems* are those partial differential equations for which the complete solution of the equation is possible with specific information at one particular instant. In reality, however, solutions to most problems require both boundary conditions and initial conditions to be specified. Mathematical modeling of real physical conditions to boundary and initial conditions for solution of partial differential equations is thus important part of the effort in obtaining the solutions.

9.2 Partial Derivatives

Mathematical formulation of partial derivatives is more complicated than for those derivatives for ordinary functions involving only one variable as defined in Section 2.2.5. The complication in expressing partial derivatives is due to the fact that the value of the function in partial derivatives is determined by variation of more than one independent variable associated with the function.

Figure 9.1 represents the derivative of a continuous ordinary function $f(x)$ with respect to its only variable x. The derivative of the function $f(x)$ described in a rectangular coordinate system is expressed in Equation 2.9 as

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{2.9}$$

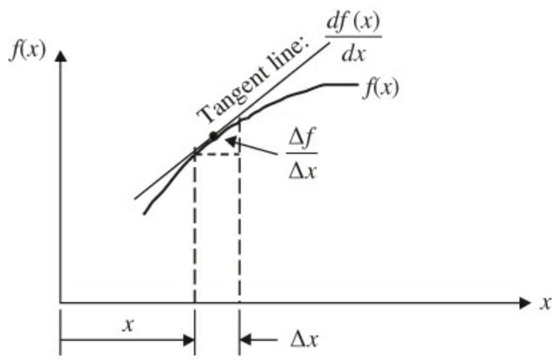


Figure 9.1 Graphic representation of an derivative of ordinary function $f(x)$.

For the function which varies with more than one independent variable—e.g., x and t expressed as $f(x,t)$—we need to express the derivative of this function with *both* of the independent variables x and t separately. The derivatives of the function $f(x,t)$ with respect to each of these two independent variables become "partial derivatives" as defined in the following.

The partial derivative of function $f(x,t)$ with respect to x only is defined as

$$\frac{\partial f(x,t)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, t) - f(x,t)}{\Delta x} \tag{9.1}$$

In this case, the independent variable t is treated as a "constant" in the derivation.

Likewise, the derivative of function $f(x,t)$ with respect to the other variable, t, is defined as

$$\frac{\partial f(x,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{f(x, t+\Delta t) - f(x,t)}{\Delta t} \tag{9.2}$$

with now the variable x being treated as a "constant" in the derivation.

The partial derivatives of higher order can be expressed in a similar way as for ordinary functions. For instance, the second-order partial derivatives of the function have the forms

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \lim_{\Delta x \to 0} \frac{\dfrac{\partial f(x+\Delta x, t)}{\partial x} - \dfrac{\partial f(x,t)}{\partial x}}{\Delta x} \tag{9.3}$$

and

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \lim_{\Delta t \to 0} \frac{\dfrac{\partial f(x, t+\Delta t)}{\partial t} - \dfrac{\partial f(x,t)}{\partial t}}{\Delta t} \tag{9.4}$$

There exists another second-order partial derivative with cross differentiations in the form

$$\frac{\partial^2 f(x,t)}{\partial x \partial t} = \frac{\partial^2 f(x,t)}{\partial t \partial x} \tag{9.5}$$

9.3 Solution Methods for Partial Differential Equations

Partial differential equations are those equations that involve partial derivatives. There are four common methods available for the solution of partial differential equations:

1. *Use appropriate "trial functions" for the solution of the equation*. Trial functions are usually in the form of polynomial function involving independent variables with unknown coefficients. These coefficients are determined in a similar way as we did for the particular solutions of second-order ordinary differential equations as described in Section 8.6. This method is rarely used in practice due to lack of a set of rules and guidelines for assuming the appropriate forms of trial functions for the solutions of partial differential equations.

2. *Use integral transforms to transform independent variables in the partial differential equations*. Such a procedure effectively converts partial differential equations into ordinary differential equations. The solution obtained from the ordinary differential equation, such as those presented in Chapters 7 and 8, needs to be inversely transformed back to the original variable domains for the complete solution. The following are three integral transform methods available for solving partial differential equations:

a. Laplace transform for variables varying in the range (0,∞).

b. Fourier transform for variables varying in the range (−∞, +∞).

c. The Hankel transform for transforming the variable in the radial coordinate (the r-coordinate) in a cylindrical polar coordinate system (r,θ,z)

The integral transform method was used to solve a variety of heat conduction problems (Ozisik 1968).

3. *Separation of variables technique*—a popular solution method for partial differential equations. The principle of this method will be described in the subsequent Section 9.3.1, with applications in the solution of partial differential equations for heat conduction in solids and vibration analysis of long cables and membranes. Not all partial differential equations are separable with respect to the involved variables, as will be demonstrated with some chapter-end problems.

4. *Numerical solution methods*—the most commonly used methods are the finite-difference (FDM) and finite-element (FEM) methods. The principle of both of these solution methods will be presented in the subsequent Chapters 10 and 11.

9.3.1 The Separation of Variables Method

The essence of this method is to "separate" the independent variables, such as x, y, z, and t involved in the functions appearing in partial differential equations.

We will illustrate the principle of this solution technique with a function $F(x,y,t)$ in a partial differential equation. The process begins with an assumption of the original function $F(x,y,t)$, to be a product of three functions, each involving only one of the three independent variables, as expressed in Equation (9.6).:

$$F(x,y,t) = f_1(x)f_2(y)f_3(t) \qquad \textbf{9.6}$$

in which $f_1(x)$, $f_2(y)$, and $f_3(t)$ are functions of the variables x, y, and t, respectively.

The expression in Equation 9.6 has effectively *separated* the three independent variables in the original function $F(x,y,t)$ into the product of three separate functions; each consists of only one of the three independent variables.

The three separate function f_1, f_2, and f_3 in Equation 9.6 will be obtained by solving three individual ordinary differential equations involving "separation constants." The solution of f_1, f_2, and f_3 from these ordinary differential equations will be related to the original function F according to what is shown in Equation 9.6, which is the solution to the original partial differential equation.

9.3.2 Laplace Transform Method for Solution of Partial Differential

Equations

The use of Laplace transform for solving partial differential equations was presented in Section 6.7 in Chapter 6, with an example on solving a partial differential equation using this technique in Example 6.18. This method is restricted to functions that are valid for the variables covering the range (0,∞). Many physical quantities that can be represented by functions involving time variable t with $t > 0$ justify the use of this solution method.

9.3.3 Fourier Transform Method for Solution of Partial Differential Equations

Like Laplace transform method, the Fourier transform is another integral transform method for solving partial differential equations in engineering analysis. The condition for using this technique in solving a partial differential equation, however, is that the variable that is transformed should cover the entire domain (−∞, ∞).

The Fourier transform for a function $f(x)$ is expressed in mathematical form as

$$\mathfrak{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = F(\omega) \tag{9.7}$$

where i is the imaginary number $\sqrt{-1}$ and ω is the transformation parameter (similar to the parameter s in the Laplace transform).

The inverse Fourier transform is obtained as Equation 9.8:

$$\mathfrak{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega \tag{9.8}$$

We recognize that the form of the Fourier transform in Equation 9.7 is similar to that of the Laplace transform in Equation (6.1), with differences in the integration limits and the exponents in the integral.

Like the Laplace transform of functions, the Fourier transform can also apply to derivatives of the functions, with variables covering the range (−∞,∞). The Fourier transform of the derivative of function $f(x)$ may be obtained from a rather simple formulae as shown in Equation 9.9:

$$\mathfrak{F}[f^n(x)] = \int_{-\infty}^{\infty} \left(\frac{d^n f(x)}{dx^n} \right) e^{-i\omega x} dx = (i\omega)^n F(\omega) \tag{9.9}$$

where n is the order of the derivatives being transformed.

We may thus derive the following expressions for the first- and second-order derivatives of function $f(x)$:

$$\mathfrak{F}[f'(x)] = i\omega F(\omega) \quad \text{and} \quad \mathfrak{F}[f''(x)] = (i\omega)^2 F(\omega) = -\omega^2 F(\omega), \quad \text{etc.} \tag{9.10}$$

Example 9.1

Find the Fourier transform of a function that is defined as

$$f(x) = \begin{cases} 0 & x < -a \\ h & -a < x < a \\ 0 & x > a \end{cases} \qquad \text{with } -\infty < x < \infty \tag{a}$$

This function is graphically represented in Figure 9.2.

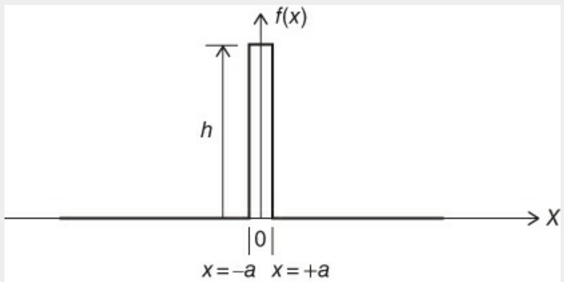


Figure 9.2 Function for Fourier Transformation.

Solution:

The Fourier transformation of the function $f(x)$ in Equation (a) may be performed by substituting the function into the definition of the Fourier transform in Equation 9.7 as follows:

$$\begin{aligned} \Im[f(x)] &= \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \\ &= \int_{-\infty}^{-a} (0)e^{-i\omega x}\, dx + \int_{-a}^{a} (h)\, e^{-i\omega x}\, dx + \int_{a}^{\infty} (0)e^{-i\omega x}\, dx = \frac{2h \sin(\omega a)}{\omega} = F(\omega) \end{aligned}$$

Table 9.1 presents a useful formulae for Fourier transforms of a some selected functions.

Table 9.1 Fourier transform of selected functions

Function for Fourier transform $f(x)$	After Fourier transform $F(\omega)$
(1) $f(x-a)$	$F(\omega)e^{-i\omega a}$
(2) $\delta(x)$*	1
(3) $u(x)$*	$(i\omega)^{-1}$
(4) $e^{-\alpha\lvert x\rvert} \quad \alpha > 0$	$\dfrac{2\alpha}{\alpha^2+\omega^2}$
(5) $u(x) \sin ax$	$\dfrac{a}{a^2-\omega^2}$
(6) $u(x) \cos ax$	$\dfrac{i\omega}{a^2-\omega^2}$

*$\delta(x)$ is the delta function, or impulse function and $u(x)$ is the unit step function. Both of these functions are defined in Section 2.4.2.

Example 9.2

Solve the following partial differential equation using Fourier transform method:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \alpha^2 \frac{\partial T(x,t)}{\partial t} \qquad -\infty < x < \infty \tag{9.11}$$

where the coefficient α is a constant. The equation satisfies the following condition:

$$T(x,t)|_{t=0} = T(x,0) = f(x) \tag{9.12}$$

Solution:

Since the variable x varies in the range $(-\infty,\infty)$, we may transform it in the function $T(x,t)$ to the parametric domain of ω via a Fourier transform as defined in Equation 9.7. Thus letting $T^*(\omega, t)$ be the Fourier-transformed function of $T(x,t)$ in Equation 9.7:

$$T^*(\omega,t) = \mathfrak{F}[T(x,t)] = \int_{-\infty}^{\infty} T(x,t)e^{-i\omega x}\,dx \tag{a}$$

Applying the above integral to the left-hand side of Equation 9.11 yields

$$\mathfrak{F}\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] = \int_{-\infty}^{\infty}\left(\frac{\partial^2 T(x,t)}{\partial x^2}\right)e^{-i\omega x}\,dx$$
$$= -\omega^2 T^*(\omega,t)$$

from the ordinary derivative in Equation 9.10.

Likewise, the Fourier transform of the right-hand side of Equation 9.11 will lead to

$$\mathfrak{F}\left[\alpha^2\frac{\partial T(x,t)}{\partial t}\right] = \alpha^2\int_{-\infty}^{\infty}\left(\frac{\partial Tx,t}{dt}\right)e^{-i\omega x}dx$$
$$= \alpha^2\frac{\partial}{\partial t}\int_{-\infty}^{\infty} T(x,t)e^{-i\omega x}\,dx$$
$$= \alpha^2\frac{\partial T^*(\omega,t)}{\partial t}$$

We have thus transformed Equation 9.11 by equating the Fourier transforms of the right-hand-side and left-hand-side terms to yield

$$-\omega^2 T^*(\omega,t) = \alpha^2\frac{dT^*(\omega,t)}{dt} \tag{b}$$

One will note that in Equation (b) after performing Fourier transform of the variable x on both sides of Equation 9.11, the partial derivatives in that equation have been replaced by the function T^* (ω, t) and an ordinary derivative of the function T^* (ω, t) on the right side of the equation.

Equation (b) is a first-order ordinary differential equation, and the method of obtaining the general solution of this equation is described in Chapter 7.

At this point, we need to transform the specified condition in Equation 9.12 by the Fourier transform defined in Equation (a), or by the expression

$$\begin{aligned} T^*(\omega, 0) &= \mathfrak{F}[T(x,0)] \\ &= \int_{-\infty}^{\infty} T(x,0)e^{-i\omega x}\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = g(\omega) \end{aligned} \quad \textbf{c}$$

in which $f(x)$ is the given condition. Thus the expression g(ω) can be obtained as the integral in Equation (c).

We will thus have the solution of the function T^* (ω, t) as

$$T^*(\omega, t) = g(\omega)e^{-(\omega^2/\alpha^2)t} \quad \textbf{d}$$

The solution of the partial differential equation in Equation 9.11 with the specified condition in Equation 9.12 may be obtained by inverting the transform T^* (ω, t) to $T(x,t)$ using Equation 9.8 through the following expression:

$$T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T^*(\omega, t)e^{i\omega x}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} [g(\omega)]e^{-(\omega^2/\alpha^2)t}e^{i\omega x}\,d\omega \quad \textbf{e}$$

where g(ω) is available in Equation (c) to be the Fourier-transformed specified condition of $T(x,0)$ in Equation 9.12.

9.4 Partial Differential Equations for Heat Conduction in Solids

9.4.1 Heat Conduction in Engineering Analysis

We have learned from Chapter 7 that temperature variations in a medium are induced by heat transmission. This variation is called the *temperature field*. Excessive heat flow can induce high-temperature fields in the medium, which may result in the following three major negative consequences to the materials:

1. High temperature weakens materials. Many properties of engineering materials change their values with temperatures. For instance, there can be significant reduction of Young's modulus of common engineering materials. Young's modulus relates to the stiffness of the material. Common sense indicates, for instance, that metals become “softer” due to such reduction of Young's modulus when heated. The ultimate strength of the material also reduces with increasing temperature. These changes have become major concerns in design analyses by engineers.

2. The basic laws of thermodynamics indicate that higher operating efficiency of thermal engines—such as internal combustion engines, gas turbines, and steam and nuclear power reactors—is achievable by operating these machines at higher temperatures. Unfortunately, high operating temperatures not only weaken the material strength as described in (1), but also introduce thermal stresses if the temperature is not uniformly distributed in machinery systems with improper mechanical constraints. These thermal stresses that are induced by heat need to be accounted for in all design analyses. Credible and reliable heat transfer analysis is thus a critical part of engineering analysis.

3. Another potentially serious concern in engineering analysis that relates to thermal effects is the possible creep deformation of structural materials. Creep is the phenomenon of continuous deformation of materials with time without them being subjected to additional loads. Creep deformation often occurs in materials at elevated temperature above half of the homologous melting point. We may thus envision that materials such as solder alloys with low melting points used in microelectronic devices are vulnerable to creep failures. The unexpected creep deformation induced by high temperature is also the source of functional problems for high-performance gas turbines and high-precision machines and devices, and is also known to be problematic for some IT (information technology) devices that require high precision in assemblies.

We thus appreciate that heat conduction analysis involving temperature fields in solid structures is an important part of engineering analysis.

9.4.2 Derivation of Partial Differential Equations for Heat Conduction Analysis

We have learned in Chapter 2 that all differential equations used in engineering analysis are derived from laws of physics, and the equations for heat conduction in solids are no exception. The law on which the derivation presented here is based is the law of conservation of energy and its derivative—the first law of thermodynamics.

Referring to Figure 9.3, a solid with a control volume is subjected to heat flow with incoming heat in the form of heat flux $\mathbf{q}(\mathbf{r},t)$ into a small element shown as the small open circle in the figure. The heat leaving the element is $\mathbf{q}(\mathbf{r}+\Delta\mathbf{r},t)$, where $\mathbf{r}$ designates the spatial variables of (x,y,z) in a rectangular coordinate system or (r,θ,z) in a cylindrical polar coordinate system.

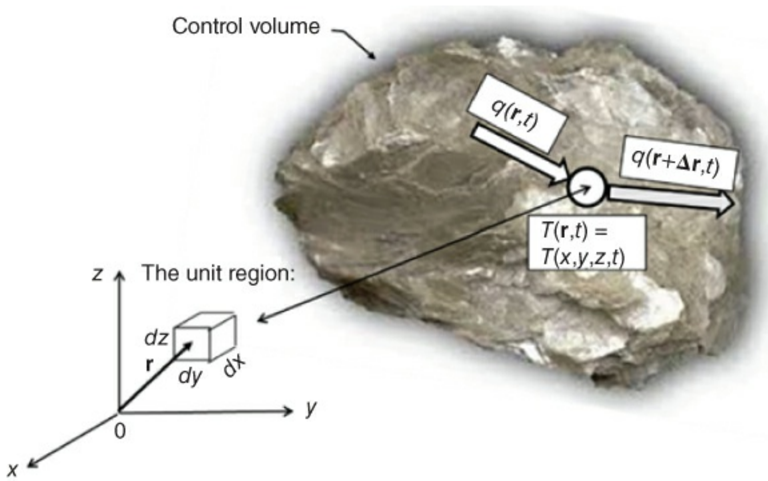


Figure 9.3 Flow of heat in a solid.

We may develop an expression for the energy balance based on the law of conservation energy as illustrated in Figure 9.4. We may use the Fourier law of heat conduction defined in Equation (7.25) to represent the heat entering and leaving the element in Figure 9.4, and the energy storage in the element may be related to temperature rise in the form of change of internal energy Δu in the solid. Mathematically, we can express Δu in the form of $\Delta u = \rho c\, \Delta T$, in which ρ is the mass density and c is the specific heat of the solid. ΔT in the expression denotes the temperature rise from its reference state. We may thus establish the following partial differential equation for the situation depicted in Figures 9.3 and the diagram in Figure 9.4:

$$\rho c \frac{\partial T(\mathbf{r},t)}{\partial t} = \nabla \cdot [k \nabla T(\mathbf{r},t)] + Q(\mathbf{r},t) \qquad \mathbf{9.13}$$

where the symbol ∇ in Equation 9.13 is the divergence defined in Section 3.5.3. $Q(\mathbf{r},t)$ is the heat generation by the solid per unit volume and time.

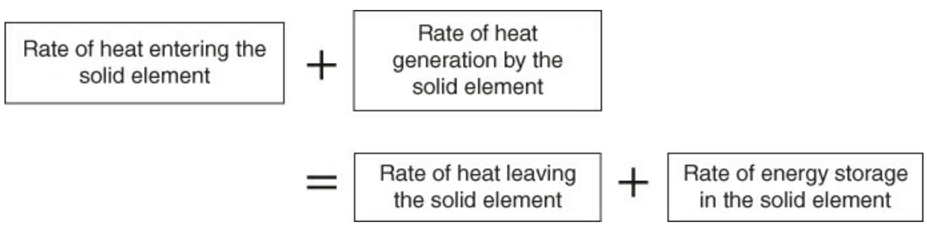


Figure 9.4 Energy balance of heat flow in a solid.

Examples of heat generation $Q(\mathbf{r},t)$ by materials might include the heat generated by the nuclear fission of uranium fuel in nuclear reactor cores, or i^2R ohmic heating of a material in electronic circuits or devices with the passage of electric current i through a resistance R.

9.4.3 Heat Conduction Equation in Rectangular Coordinate Systems

The general heat conduction equation in Equation 9.13 will take the following form with $T(\mathbf{r},t) = T(x,y,z,t)$:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y \frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] + Q(x,y,z,t) \qquad \mathbf{9.14a}$$

in which k_x, k_y, and k_z are the thermal conductivities of the solid along the x-, y-, and z-coordinate, respectively.

9.4.4 Heat Conduction Equation in a Cylindrical Polar Coordinate System

Heat conduction equation in this coordinate system is obtained by expanding Equation 9.13 as follows with $T(\mathbf{r},t) = \mathrm{T}(r,\theta,z,\mathrm{t})$:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial r}\left[k_r \frac{\partial T}{\partial r}\right] + \frac{1}{r}\left[k_r \frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\left[\frac{\partial}{\partial \theta} k_\theta \frac{\partial T}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] + Q(r,\theta,z,t) \qquad \textbf{9.14b}$$

where k_r, k_θ, and k_z are thermal conductivities of the material along the r-, θ- and z-coordinates respectively.

9.4.5 General Heat Conduction Equation

Thermal conductivities k_x, k_y, and k_z in Equation 9.14a and k_r, k_θ, and k_z in Equation 9.14b are used for heat conduction analysis of solids with their thermophysical properties varying in different directions, such as for fiber filament composites. For most engineering analyses, such variations of thermophysical properties do not exist. Consequently, a generalize heat conduction equation may be expressed as follows:

$$\nabla^2 T(\mathbf{r},t) + \frac{Q(\mathbf{r},t)}{k} + \frac{1}{\alpha}\frac{\partial T(\mathbf{r},t)}{\partial t} \qquad \textbf{9.15}$$

where k = thermal conductivity of the material and $Q(\mathbf{r},t)$ is the heat generated by the material per unit volume and time.

The symbol α in Equation 9.15 is the "thermal diffusivity" of the material, its value being $\alpha = k/\rho c$; it is often used as a measure of how "fast" heat can flow by conduction in a solid.

9.4.6 Initial and Boundary Conditions

Solution of the heat conduction equation in Equation 9.15 involves determining a number of arbitrary constants according to specific initial and boundary conditions. These conditions are necessary to translate the real physical conditions into mathematical expressions. Proper specification and translation of these conditions are important steps in formulating numerical solutions, either by the finite-difference method or by the finite-element method, as will be described in Chapters 10 and 11, respectively.

Initial conditions are required only when dealing with transient heat transfer problems in which the temperature field in a solid changes with the elapsing time. These conditions specify the temperature distribution in the solid before and at the onset of the changing thermal conditions that create temperature distributions. The common initial condition in a solid can be expressed mathematically as

$$T(\mathbf{r},t)|_{t=0} = T(\mathbf{r},0) = T_0(\mathbf{r}) \qquad \textbf{9.16}$$

where the temperature field $T_0(\mathbf{r})$ is a function of the spatial coordinates $\mathbf{r}$ only.

In many practical applications, the initial temperature distribution $T_0(\mathbf{r})$ in Equation 9.16 can be assigned a constant value such as room temperature at 20°C for uniform temperature (isothermal) conditions.

Specific boundary conditions are required, however, in the analysis of all transient and steady-state problems involving solids of finite shape. Several types of boundary conditions are commonly used, as will be described below.

a. *Prescribed surface temperature,* $T_s(t)$. Quite often, in practice, the temperature at the surface of the solid structure is measured by either attaching thermocouples to the structure surface or by some noncontact methods such as infrared thermal imaging scanning camera. The mathematical expression for this case takes the form

$$T(\mathbf{r},t)|_{\mathbf{r}=\mathbf{r}_s} = T_s(t) \qquad \textbf{9.17a}$$

where $\mathbf{r}_s$ is the coordinates of the boundary surface where temperature is specified as $T_s(t)$. This type of boundary condition with prescribed surface temperatures is called a Dirichlet condition by mathematicians.

b. *Prescribed heat flux boundary condition,* $q_s(t)$. Many structures have their surfaces exposed to a heat source or a heat sink. One such example is the heat treatment of a large forged piece, for example, a turbine shaft in an autoclave in which heat is being supplied to the piece through its outside surface. The mathematical translation of the heat flux to or from a solid surface can be readily carried out by using the Fourier law of heat conduction as defined in Equation (7.25). The mathematical formulation of the heat flux across a solid boundary surface can be expressed as

$$\left.\frac{\partial T(\mathbf{r},t)}{\partial \mathbf{n}_i}\right|_{\mathbf{r}=\mathbf{r}_s} = -\frac{q_s(\mathbf{r}_s,t)}{k} \qquad \textbf{9.17b}$$

where k is the thermal conductivity of the solid material. The symbol $\partial/\partial\mathbf{n}_i$ is the differentiation along the outward-drawn normal to the boundary surface S_i. The term $q_s(\mathbf{r}_s,t)$ in Equation 9.17b is the specified heat flux across this surface defined by the coordinate $\mathbf{r}_s$ in the same direction as $\mathbf{n}_i$. This type of boundary condition is referred to as the Neuman condition by mathematicians. We may express Equation 9.17b for the boundaries that are impermeable to heat flow, or a boundary that is thermally insulated as

$$\left.\frac{\partial T(\bar{\mathbf{r}},t)}{\partial \mathbf{n}}\right|_{\mathbf{r}=r_s} = 0 \qquad \textbf{9.17c}$$

c. *Convective boundary conditions*. Many engineering applications involve boundary surfaces of a solid in contact with fluids in gaseous or liquid states such as described in Section 7.5.5. A special expression has to be derived for the boundary conditions of this type.

One common phenomenon occurs when a solid is submerged in a stationary or moving fluid at a different temperature; a boundary layer is developed at the interface of the solid and fluid as illustrated in Figure 9.5. This boundary layer acts as a "barrier" for heat transfer between the solid and the surrounding fluid, resulting in $T_s \neq T_f$, where T_s is the temperature at the surface of the solid and T_f is the contacting bulk fluid temperature. The thickness of the boundary layer is related to the temperature difference between the two media and the velocity of the moving fluid in contact in the case of forced convective heat transfer in the surrounding fluid. The resistance induced by the boundary layer is related to the heat transfer coefficient h as described in Sections 7.5.4 and 7.5.5.

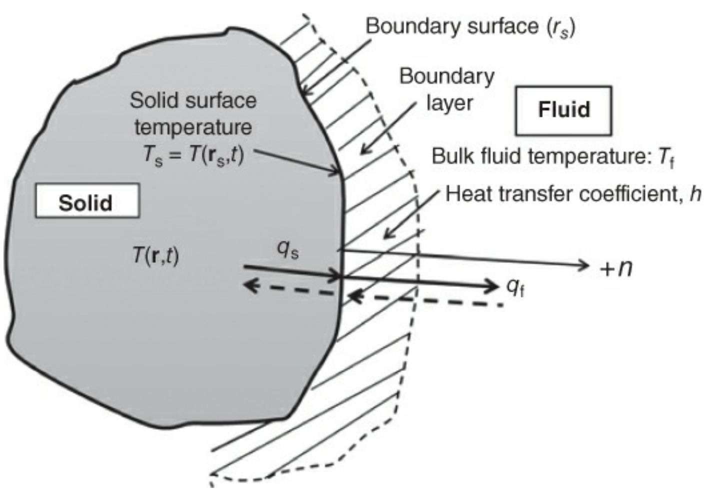


Figure 9.5 Heat transfer at the interface of a solid submerged in fluid.

The mathematical expression for the boundary condition for the solid submerged in fluid can be derived by equating the heat flux q_s entering the boundary surface $\mathbf{r}_s$ of the solid with the heat conduction in the solid and the heat flux q_f leaving the same boundary surface by convective heat transfer. Mathematical expressions for heat conduction follow Fourier's law of heat conduction and the formula for convective heat transfer is expressed by Newton' cooling law in Section 7.5.4. We thus have the equality

$$-k\frac{\partial T(\mathbf{r},t)}{\partial n}\bigg|_{\mathbf{r}=\mathbf{r}_s} = h[T(\mathbf{r}_s,t) - T_f]$$

which leads to the following boundary condition for heat conduction analysis of the solid:

$$\frac{\partial T(\mathbf{r},t)}{\partial n}\bigg|_{\mathbf{r}=r_s} + \frac{h}{k}T(\mathbf{r},t)|_{\mathbf{r}=\mathbf{r}_s} = \frac{h}{k}T_f \qquad \textbf{9.17d}$$

where h is the heat transfer coefficient of the surrounding fluid and k is the thermal conductivity of the solid.

The "mixed boundary condition" expressed in Equation 9.17d actually could be used for problems involving prescribed surface temperatures (or Dirichlet conditions) in Equation 9.17a with $h \to \infty$, which makes $T(\mathbf{r},t)|_{\mathbf{r}=\mathbf{r}_s} = T_s = T_f$: that is, the surface temperature of the solid (T_s) is equal to the bulk fluid temperature of the surrounding fluid (T_f). We may also show that letting $h = 0$ in Equation 9.17d will lead to a thermally insulated boundary condition with $q_s = 0$ in Equation 9.17b.

Example 9.3

Find the boundary conditions of a long, thick-walled pipe for hot steam flow at a bulk temperature T_s and heat transfer coefficient h_s. The outside wall of the pipe is in contact with cold air at a temperature of T_a and a heat transfer coefficient h_a, as illustrated in Figure 9.6.

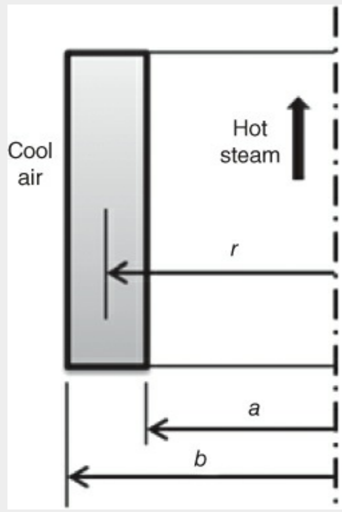


Figure 9.6 Steam flow in a thick wall pipe.

Solution:

We may hypothesize that heat will flow in the positive radial direction (r) in a long pipe such as in this example because the temperature gradient crosses the pipe wall. We thus recognize two boundary surfaces in this example at the inner surface with $r = a$ and the outside surface at $r = b$.

We may use Equation 9.17d to establish the conditions at both boundary surfaces as follows.

At inner boundary with r = a:

$$k\left.\frac{dT(r)}{dr}\right|_{r=a} - \frac{h_s}{k}T(r)\Big|_{r=a} = \frac{h_s}{k}T_s$$

At the outside boundary with r = b:

$$k\left.\frac{dT(r)}{dr}\right|_{r=b} + \frac{h_a}{k}T(r)|_{r=b} = \frac{h_a}{k}T_a$$

In the above expressions, k is the thermal conductivity of the pipe material.

Example 9.4

Find the temperature distribution in a long, thick-walled pipe with inner and outside radii *a* and *b*, respectively, by using the three types of boundary conditions in Equations 9.17b, and 9.17d. Conditions for establishing the mathematical expressions for these boundary conditions with hot steam inside the pipe and cool surrounding air outside the pipe are indicated in Figure 9.7.

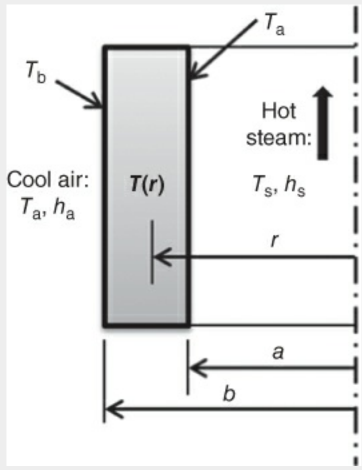


Figure 9.7 Temperature variations in the pipe wall.

Solution:

We recognize that the physical situation of the current example is the same as that in Example 9.3: that the principal direction of heat flow is from the hot steam inside the pipe to the outside cool air in the radial direction. We may thus hypothesize that the temperature distribution in the pipe wall is represented by $T(r)$ as illustrated in Figure 9.7.

We may select the relevant terms from the heat conduction equation derived for the cylindrical polar coordinate system expressed in Equation 9.14b as follows:

$$\frac{d^2T(r)}{dr^2} + \frac{1}{r}\frac{dT(r)}{dr} = 0 \quad \textbf{a}$$

Equation (a) is a nonlinear second-order homogeneous differential equation. One may obtain the solution of this equation by expressing Equation (a) in the following form:

$$\frac{d}{dr}\left[r\frac{dT(r)}{dr}\right] = 0 \quad \textbf{b}$$

The solution of temperature distribution $T(r)$ in Equation (b) may be obtained by integrating Equation (b) twice and result in the following solution:

$$T(r) = c_1 \ln(r) + c_2 \quad \textbf{c}$$

The two arbitrary constants c_1 and c_2 in Equation (c) can be determined by three different types of boundary conditions given in Equations 9.17b, and 9.17d.

With prescribed boundary conditions in Equation 9.17a We have T_a as the temperature at the inner surface with $T(a) = T_a$, and $T(b) = T_b$ at the outside surface of the pipe.

Substituting the above boundary conditions into Equation (c), we have

$$c = \frac{T_a - T_b}{\ln(a/b)} \quad \text{and} \quad c_2 = T_a - \frac{T_a - T_b}{\ln(a/b)} \ln(a)$$

The temperature distribution in the pipe wall is thus obtained as

$$T(r) = T - \frac{T_a - T_b}{\ln(a/b)} \ln\left(\frac{r}{a}\right) \tag{d}$$

With prescribed heat flux *q*a across the inner surface and *T*b at the outside surface We need to solve the following differential equation with the aforementioned boundary conditions:

$$\frac{d^2T(r)}{dr^2} + \frac{1}{r}\frac{dT(r)}{dr} = 0 \tag{a}$$

with the following boundary conditions according to Equation 9.17b:

$$\left.\frac{dT(r)}{dr}\right|_{r=a} = -\frac{q_a}{k} \tag{e}$$

where q_a is the specific heat flux input to the inner surface of the pipe wall.

The boundary condition at the outside surface of the pipe is established by using Equation 9.17a as

$$T(r)|_{r=b} = T(b) = T_b \tag{f}$$

We may determine the constants c_1 and c_2 by substituting the boundary conditions in Equations (e) and (f) into Equation (c):

$$c = -\frac{aq_a}{k} \quad \text{and} \quad c = T + \frac{aq_a}{k} \ln(b)$$

The temperature distribution in the pipe wall $T(r)$ can thus be obtained by substituting the constants c_1 and c_2 in the above expressions into Equation (c), resulting in the following solution:

$$T(r) = T - \frac{aq_a}{k} \ln\left(\frac{r}{b}\right) \tag{g}$$

With mixed boundary conditions Referring to Figure 9.7 and Equation 9.17d, the two boundary conditions for this case are

$$\left.\frac{dT(r)}{dr}\right|_{r=a} - \frac{h_s}{k} T(r)|_{r=a} = \frac{h_s}{k} T_s \tag{h}$$

$$\left.\frac{dT(r)}{dr}\right|_{r=b} + \frac{h_a}{k} T(r)|_{r=b} = \frac{h_a}{k} T_a \tag{j}$$

The constants c_1 and c_2 in the solution of Equation (a) may be determined by applying the boundary conditions in Equations (h) and (j), resulting in

$$c_1 = \frac{h_s h_a (T_a - T_s)}{\dfrac{kh_a}{a} + \dfrac{kh_s}{b} + h_s h_a \ln\left(\dfrac{b}{a}\right)}$$

and

$$c_2 = T_a - \frac{h_s h_a (T_a - T_s)}{\frac{kh_a}{a} + \frac{kh_s}{b} + h_s h_a \ln\left(\frac{b}{a}\right)} \left(\frac{k}{h_a b} + \ln(b)\right)$$

The temperature distribution in the pipe wall may be obtained by substituting the constants c_1 and c_2 in the above expressions into the solution in Equation (c).

9.5 Solution of Partial Differential Equations for Transient Heat Conduction Analysis

The partial differential equation in Equation 9.15 is valid for the general case of heat conduction in solids including transient cases in which the induced temperature field $T(r,t)$ varies with time t. In this section, we will demonstrate how the separation of variables technique described in Section 9.3 can be used to solve problems of this type in both rectangular and cylindrical polar coordinate systems.

9.5.1 Transient Heat Conduction Analysis in Rectangular Coordinate System

The case that we will present here is a large, flat slab with thermal conductivity k. The slab has thickness L as illustrated in Figure 9.8. It has an initial temperature given by a specified function $f(x)$, and the temperatures of both its faces are maintained at temperature T_f at time $t > 0$. We need to determine the temperature variation in the slab with respect to time t.

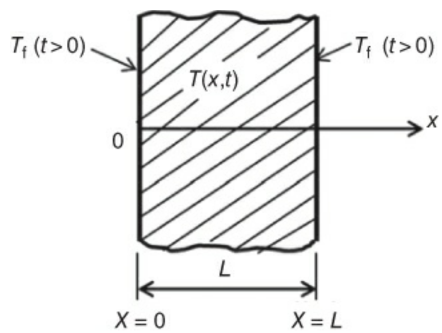


Figure 9.8 Conduction of heat across thickness of a slab.

We recognize that the physical situation of this problem is relevant to the cooling of a slab from its initial temperature distribution according to an initial temperature function $f(x)$ to a new state in which both its faces are exposed to a uniform temperature T_f for $t > 0$. It is thus clear that the heat flow will be along the x-coordinate—that is, the transient temperature in the slab may be expressed as $T(x,t)$.

One may imagine that the temperature in the slab will vary continuously with time t, until the temperature in the slab reaches a uniform temperature T_f. The purpose of our subsequent analysis, however, is to find the transient temperature $T(x,t)$ in the slab before it reaches the ultimate uniform temperature of T_f.

The governing differential equation for this physical situation may be deduced from Equations 9.14a and 9.15 with the thermal conductivity of the slab material $k_x = k_y = k_z = k$ for an isotropic material. The terms $Q(x,y,z,t)$ in Equation 9.14a and $Q(\mathbf{r},t)$ in Equation 9.15 are omitted because the slab does not generate heat by itself. Consequently, the equation that describes the specified physical situation becomes

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \qquad \textbf{9.18}$$

where α is the thermal diffusivity of the slab material.

The solution of Equation 9.18 will satisfy the following specified conditions:

Initial condition (IC):

$$T(x,t)|_{t=0} = T(x,0) = f(x) \quad \text{a given function} \qquad \textbf{9.19a}$$

Boundary conditions (BCs):

$$T(x,t)|_{x=0} = T(0,t) = T_f \qquad t > 0 \tag{9.19b}$$

$$T(x,t)|_{x=L} = T(L,t) = T_f \qquad t > 0 \tag{9.19c}$$

Equations 9.18 and Equations 9.19b, and 9.19c are proper mathematical expressions for the problem. However, the solution of Equation 9.18 may be simplified by converting the nonhomogeneous BCs in Equations 9.19b and 9.19c to homogeneous ones by the substitution of $u(x,t)$ for $T(x,t)$:

$$u(x,t) = T(x,t) - T_f \tag{9.20}$$

Equations 9.18 and 9.19 will thus be converted to the following equivalent system of differential equations with modified IC and BCs by the substitution of Equation 9.20 into Equations 9.18 and Equations 9.19b, and 9.19c:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} \tag{9.21}$$

with the following converted initial condition:

$$u(x,t)|_{t=0} = u(x,0) = f(x) - T_f \tag{a}$$

and for the BCs:

$$u(x,t)|_{x=0} = u(0,t) = T(x,t)|_{x=0} - T_f = T_f - T_f = 0 \tag{b}$$

$$u(x,t)|_{x=L} = u(L,t) = T(x,t)|_{x=L} - T_f = T_f - T_f = 0 \tag{c}$$

The solution of Equation 9.21 satisfying the IC and BCs in Equations (a), (b), and (c) can be obtained using either the Laplace transform with t as the variable in the transformation, as presented in Section 6.7, or using the separation of variables technique as described in Section 9.3.1. We will use the latter technique for obtaining the solution of Equation 9.21.

The solution process begins with the separation of the variables in the function for the solution. In the present case, the variables to be separated are variables x and t in the function $u(x,t)$. We will proceed by letting

$$u(x,t) = X(x)\tau(t) \tag{9.22}$$

in which function $X(x)$ is a function with variable x only and the other function $\tau(t)$ has a variable t only.

Substituting the relationship in Equation 9.22 into Equation 9.21 gives the expression

$$\frac{\partial^2[X(x)\tau(t)]}{\partial x^2} = \frac{1}{\alpha}\frac{\partial[X(x)\tau(t)]}{\partial t}$$

The above expression leads to the following equalities:

$$\tau(t)\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{\alpha}X(x)\frac{\partial \tau(t)}{\partial t} \tag{d}$$

or

$$\tau(t)\frac{d^2 X(x)}{dx^2} = \frac{1}{\alpha}X(x)\frac{d\tau(t)}{dt}$$

The second expression in Equation (d) will yield the following special form of equality:

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = \frac{1}{\alpha}\frac{1}{\tau(t)}\frac{d\tau(t)}{dt}$$

We observe that the left-hand side (LHS) of the expression contains only variable x, whereas the right-hand side (RHS) of the expression consists of another independent variable t. The only way this type of equality can exist is to have LHS = RHS = constant, and this constant needs to be a negative number to be meaningful in the solution. Consequently, we will introduce a "separation constant" $-\beta^2$ for the above expression as follows:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{\alpha}\frac{1}{\tau(t)}\frac{d\tau(t)}{dt} = -\beta^2 \tag{9.23}$$

One may derive two separate ordinary differential equations by equating the LHS = $-\beta^2$ in Equation 9.23 to give

$$\frac{d^2X(x)}{dx^2} + \beta^2 X(x) = 0 \tag{9.24}$$

and another differential equation by letting the terms in the RHS = $-\beta^2$ in Equation 9.23 to give to attend these first two classes

$$\frac{d\tau(t)}{dt} + \alpha\beta^2\tau(t) = 0 \tag{9.25}$$

We have thus successfully separated the variables x and t from the function $u(x,t)$ in Equation 9.21 into two ordinary differential equations in Equations 9.24 and 9.25 for the solutions of functions $X(x)$ and $\tau(t)$ in Equation 9.22.

The solution $X(x)$ and $\tau(t)$ in Equations 9.24 and 9.25 requires the specific condition for both these equations. We will derive these conditions for Equation 9.24 by the following procedures.

Let us first substitute the expression $u(0,t) = 0$ in equation (b) into Equation 9.22, resulting in: $u(0,t) = X(0)\tau(t) = 0$. We see that function $\tau(t) \neq 0$; we will thus have $X(0) = 0$. The condition $X(L) = 0$ can be derived in a similar way. We will thus have the specified conditions for Equation 9.24:

$$X(0) = 0 \tag{e1}$$

$$X(L) = 0 \tag{e2}$$

Equation 9.24 is a linear homogeneous second-order differential equation, and the solution $X(x)$ is available from Section 8.2 in the form

$$X(x) = A\cos\beta x + B\sin\beta x \tag{f}$$

in which A and B are arbitrary constants, and β is the unknown separation constant. We will use the specified condition in Equation (e1) to determine $A = 0$, and the condition in Equation (e2) leads to $X(L) = B\ \sin\ \beta L = 0$. The latter expression provides two possibilities, either the constant $B = 0$ or $\sin \beta L = 0$. Since the constant A in the solution of $X(x)$ is already equal to zero, having $B = 0$ will lead to $X(x) = 0$, and therefore $u(x,t) = 0$ according to Equation 9.22. The solution $u(x,t) = 0$ is a trivial solution of Equation 9.22 and it is considered to be unrealistic. We are thus left with the only possibility of letting

$$\sin\beta L = 0 \tag{9.26}$$

We will quickly realize that there are multiple values of the separation constant β that satisfy Equation 9.26. These are $\beta = n\pi$, with $n = 1, 2, 3, \ldots$. Alternatively, we may express the separation constant in the form

$$\beta_n = \frac{n\pi}{L} \qquad (n = 1, 2, 3, \ldots) \tag{9.27}$$

We may thus express the solution of Equation 9.24, that is, the function $X(x)$, in the following form:

$$X(x) = B_1 \sin\frac{\pi x}{L} + B_2 \sin\frac{2\pi x}{L} + B_3 \sin\frac{3\pi x}{L} + \cdots$$
$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad (n = 1, 2, 3, \ldots) \tag{9.28}$$

In Equation 9.28, the constant coefficients B_1, B_2, B_3, ... correspond to the constant coefficient B in the solution of $X(x)$ in Equation (f). We need to include all valid solutions of $X(x)$ in the solutions of linear differential equation expressed in Equation 9.24, with those expressed in Equation 9.28.

We will note that the separation constant β in Equation 9.27 has multiple values excluding the case of $n = 0$, and all other values are multiples of integer numbers. These facts make $\sin \beta L = 0$ in Equation 9.26 the "characteristic equation" of the differential equation in Equation 9.24, and the solutions $\beta L = n\pi$ ($n = 1, 2, 3, \ldots$) the eigenvalues as defined in Section 4.8.

We are now ready to solve for the other function $\tau(t)$ in Equation 9.25. That equation is a first-order differential equation and the solution $\tau(t)$ may be found by the solution methods presented in Section 7.2. Thus, we will have the solution for Equation 9.25 in the following form:

$$\tau(t) = C_n e^{-\alpha \beta_n^2 t} \tag{9.29}$$

where C_n with $n = 1, 2, 3, \ldots$ are multi-valued integration constants because of the multi-valued β_n in the solution.

The general solution of Equation 9.21 can thus be obtained by substituting the solutions $X(x)$ in Equation 9.28 and $\tau(t)$ in Equation 9.29 into Equation 9.22 to give

$$u(x,t) = \sum_{n=1}^{\infty} C_n B_n e^{-\alpha\beta_b t} \sin\frac{n\pi x}{L}$$
$$= \sum_{n=1}^{\infty} b_n e^{-\alpha\beta_n t} \sin\frac{n\pi x}{L} \tag{9.30}$$

We have lumped the product of constants $C_n B_n$ and make this product of multi-valued constants to a single multi-valued constant b_n in Equation 9.30. The constants b_n ($n = 1, 2, 3, \ldots$) may be determined by the remaining specified condition in Equation (a) with $u(x,0) = f(x) - T_f$:

$$u(x,0) = f(x) - T_f = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{9.31}$$

Constants b_n may be determined by the following steps involving the use of the orthogonality property of integrals of trigonometric functions as follows.

Step 1: Multiply both sides of Equation 9.31 with the function $\sin(n\pi x/L)$:

$$\left(\sin\frac{n\pi x}{L}\right)[f(x) - T_f] = \left(\sin\frac{n\pi x}{L}\right)\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$
$$= \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L} \tag{g}$$

Step 2: Integrate both sides of Equation (g) with integration limits of $(0, L)$:

$$\int_0^L \left(\sin\frac{n\pi x}{L}\right)[f(x) - T_f]\,dx = \int_0^L \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L}\,dx$$
$$= \sum_{n=1}^{\infty}\int_0^L b_n\left(\sin\frac{n\pi x}{L}\right)^2 dx \quad \textbf{h}$$

Step 3: Make use of the orthogonality of the harmonic functions such as sine and cosine functions through the following relationships:

$$\int_0^p \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ p/2 & \text{if } m = n \end{cases} \quad \textbf{9.32}$$

which imply that

$$\int_0^p \left(\sin\frac{n\pi x}{p}\right)^2 dx = \frac{p}{2}$$

with $m = n$ in Equation 9.32. We thus have from Equation (h)

$$\int_0^L \left(\sin\frac{n\pi x}{L}\right)[f(x) - T_f]\,dx = b_n\left(\frac{L}{2}\right),$$

which leads to the unknown coefficients b_n being

$$b_n = \frac{2}{L}\int_0^L [f(x) - T_f]\sin\frac{n\pi x}{L}\,dx \quad \textbf{9.33}$$

We thus have the solution of Equation 9.21 as

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L [f(x) - T_f]\sin\frac{n\pi x}{L}\,dx\right\}e^{-\alpha\beta_n^2 t}\sin\frac{n\pi x}{L}$$

The solution of $T(x,t)$ in Equation 9.18 for the temperature distribution in the slab can thus be obtained by the relationship in Equation 9.20 to take the form

$$T(x,t) = T_f + \frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L [f(x) - T_f]\sin\frac{n\pi x}{L}\,dx\right\}e^{-\alpha\beta_n^2 t}\sin\frac{n\pi x}{L} \quad \textbf{9.34}$$

where T_f is the temperature of both surfaces in Figure 9.8, L is the thickness of the slab, $f(x)$ represents the initial temperature distribution in the slab, and $\beta_n = n\pi/L$ with $n = 1, 2, 3, \ldots$ are the separation constants.

One would envisage that the slab, with the initial temperature distribution $T(x,t)|_{t=0} = f(x)$, would after having both its surfaces maintained at temperature T_f for time $t > 0$ ultimately reach a uniform temperature T_f. This solution reflects such a physical situation with $t \to \infty$ for which $T(x,\infty) = T_f$ will be obtained as the solution of Equation 9.34.

9.5.2 Transient Heat Conduction Analysis in the Cylindrical Polar Coordinate System

We will present the next case of solving heat conduction problems using the separation of variables technique by replacing the flat slab in Section 9.5.1 with a solid cylinder of radius a as shown in Figure 9.9. The cylinder initially has a temperature distribution of $f(r)$. It is submerged in an agitated fluid with bulk fluid temperature T_f. Because of the vigorous agitation of the fast-moving surrounding fluid, the heat transfer coefficient h approaches ∞, resulting in a circumference temperature of the cylinder being the same as the surrounding bulk fluid temperature T_f for time t > 0 as in Equation 9.17d. The initial temperature distribution in the cylinder $T(r)$ will thus vary with time t after it is submerged in the fluid at t > 0. We need to find the temperature distribution $T(r,t)$ in the cylinder for t > 0.

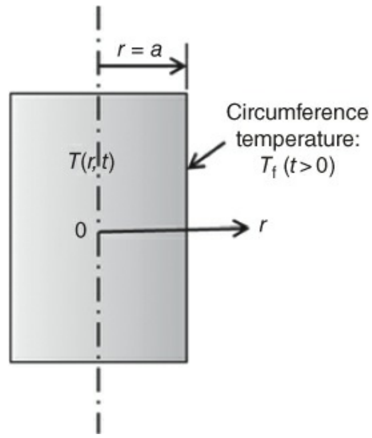


Figure 9.9 Heat Conduction in a Long Solid Cylinder.

The reader may relate the current case to a common industrial application in the quenching of hot solids in a cold fluid bath. We assume that the cylinder is long enough to justify a simplified case of having the principal heat flow in the radial direction (r).

The applicable partial differential equation for the current application may be deduced from Equation 9.14b by dropping the third and other terms in the right-hand side of the equation, resulting in

$$\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} = \frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,t)}{\partial r} \qquad \textbf{9.35}$$

where $\alpha = k/\rho c$ is the thermal diffusivity of the cylinder material, with ρ and c being the mass density and specific heat of the cylinder material, respectively.

The appropriate conditions for the current problem are

Initial condition:

$$T(r,t)|_{t=0} = T(r,0) = f(r) \qquad \textbf{a}$$

The function $f(r)$ in Equation (a) is the specified temperature distribution in the radial direction at time t = 0.

Boundary conditions:

$$T(r,t)|_{r=a} = T(a,t) = T_f \qquad t > 0 \qquad \textbf{b1}$$

where T_f is the given bulk fluid temperature.

The other boundary condition at r = 0, the center of the solid cylinder, has to be a finite value. The mathematical expression of this condition is

$$T(r,t)|_{r=0} = T(0,t) \neq \infty \quad \text{or} \quad T(0,t) = \text{finite value} \qquad \textbf{b2}$$

We will convert Equation 9.35 from $T(r,t)$ to $u(r,t)$ as we did in Section 9.5.1. We thus have

$$u(r,t) = T(r,t) - T_f \qquad \textbf{c}$$

Substituting the above expression into Equation 9.35 and the associated initial and boundary conditions in Equations (a) and (b), we obtain the following converted partial differential equation

$$\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} \tag{9.36}$$

with the converted specified conditions:

$$u(r,t)|_{t=0} = u(r,0) = f(r) - T_f \quad \text{for } t = 0 \tag{d}$$

$$u(r,t)|_{r=a} = u(a,t) = 0 \quad \text{for } t > 0 \tag{e}$$

We may solve Equation 9.36 using either the Laplace transform method with variable t being the transform parameter according to the procedure presented in Section 6.7, or we may use the separation of variables technique as we did in Section 9.3.1. We will learn to use the latter technique for solving partial differential equations in the cylindrical polar coordinate system.

Letting the solution of function $u(x,t)$ in Equation 9.36 with two independent variables r and t be separated according to the following relation:

$$u(r,t) = R(r)\tau(t) \tag{9.37}$$

where the function $R(r)$ is a function of variable r only and the function $\tau(t)$ is a function of the other variable t only. Upon substituting this relation into Equation 9.36 we get

$$\frac{1}{\alpha}\frac{\partial[R(r)\tau(t)]}{\partial t} = \frac{\partial^2[R(r)\tau(t)]}{\partial r^2} + \frac{1}{r}\frac{\partial[R(r)\tau(t)]}{\partial r} \tag{f}$$

or

$$\frac{R(r)}{\alpha}\frac{\partial\tau(t)}{\partial t} = \tau(t)\frac{\partial^2 R(r)}{\partial r^2} + \frac{\tau(t)}{r}\frac{\partial R(r)}{\partial r}$$

Equation (f) legitimates the conversion of the partial derivatives of $R(r)$ and $\tau(t)$ to ordinary derivatives because both these functions involve only one variable. Consequently, we may establish the following equality as:

$$\frac{R(r)}{\alpha}\frac{d\tau(t)}{dt} = \tau(t)\frac{d^2 R(r)}{dr^2} + \frac{\tau(t)}{r}\frac{dR(r)}{dr}$$

After rearranging the terms in the above expression, we get the following equality:

$$\frac{1}{\alpha\tau(t)}\frac{d\tau(t)}{dt} = \frac{1}{R(r)}\left[\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr}\right] \tag{g}$$

We note that the left-hand side of Equation (g) is a function of variable t only and the right-hand side involves another function of the other variable r only. The only way this equality can hold is to have both sides equal to a constant (a negative constant to be realistic). Thus we have the following expression:

$$\frac{1}{\alpha\tau(t)}\frac{d\tau(t)}{dt} = \frac{1}{R(r)}\left[\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr}\right] = -\beta^2 \tag{9.38}$$

Equation 9.38 will result in two separate ordinary differential equations:

$$\frac{d\tau(t)}{dt} + \alpha\beta^2\tau(t) = 0 \tag{9.39}$$

and

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + \beta^2 r^2 R(r) = 0 \tag{9.40}$$

The solution of Equation 9.39 is identical to Equation 9.29 in the form

$$\tau(t) = c_n e^{-\alpha \beta_b^2 t} \tag{h}$$

where c_n with n = 1, 2, 3,..... is a multi-valued integration constant.

Equation 9.40 is a homogeneous second-order ordinary differential equation but without constant coefficients as in the cases in Section 8.2. Therefore the solution method presented in that section cannot be used for the solution of Equation 9.40. However, a close look at this equation reveals that it is a special case of the Bessel equation in Equation (2.27) with order n = 0. Consequently, the solution of Equation 9.40 can be expressed by the Bessel functions given in Equation (2.28) with n = 0 in the following form:

$$R(r) = AJ_0(\beta r) + BY_0(\beta r) \tag{9.41}$$

where A and B are arbitrary constants to be determined by appropriate boundary conditions, and $J_0(\beta r)$ and $Y_0(\beta r)$ are the Bessel functions of first and second kind with order zero, respectively.

The two constants A and B in Equation 9.41 can be determined using the two boundary conditions specified in Equations (b1) for $r = a$, and Equation (b2) for r = 0. Application of the latter condition for $T(0,t)$ to be finite results in B = 0 for the reason that the Bessel function of second kind $Y_0(\beta r)|_{r=0} = Y(0) \to -\infty$, which violates the condition that the value of $T(x,t)$ cannot be $\pm\infty$. Consequently, the only way we can avoid $T(0,t) \to -\infty$ is to let the constant B = 0 in Equation 9.41.

We thus have the solution of Equation 9.40 as

$$R(r) = AJ_0(\beta r) \tag{j}$$

The remaining arbitrary constant A in the solution in Equation (j) can be evaluated by the boundary condition in Equation (e) that

$$R(a) = AJ_0(\beta a) = 0$$

The above expression offers two possible solutions: we may either let the constant A = 0 or have $J_0(\beta a)$ = 0. We recall that constant B in the expression for $R(r)$ in Equation 9.41 is already set to zero. Setting the other constant A = 0 will make $R(r)$ and thus $u(r,t)$ and $T(r,t)$ zero in the cylinder at all times, which violates physical reality. Consequently, we need to let

$$J_0(\beta a) = 0 \tag{9.42}$$

Equation 9.42 offers the values of the separation constant β in Equation 9.38 because $J_0(x)$ = 0 is an equation that has multiple roots like sin(βL) = 0 in Equation 9.26. The roots of the equation $J_0(\beta a)$ = 0 may be found either from Figure 2.45a or from mathematical handbooks. For instance, the first six approximate solutions for βa in Equation 9.42 can be found to be 2.4, 5.52, 8.65, 11.79, 14.93, and 18.07 (Zwillinger, 2003), or from commercial software such as “Mathematica” or “MatLAB” as will be described in Chapter 10.

Example 9.5

Find the first four roots of Equation 9.42 with a = 20 units.

Solution:

We are required to find the first four roots of the equation of $J_0(20\beta) = 0$.

We will find the first four approximate roots of the equation $J_0(x) = 0$ to be x = 2.4, 5.52, 8.65, and 11.79 (Zwillinger, 2003). This equation may be related to the equation $J_0(20\beta) = 0$ with $x = 20\beta$. We will thus have the first four roots of the equation $J_0(20\beta) = 0$ with $\beta = x/20$, with β_1 = 2.4/20 = 0.12, β_2 = 5.52/20 = 0.276, β_3 = 8.65/20 = 0.4325, and β_4 = 11.79/20 = 0.5895.

We may thus express the separation variable β in Equation 9.38 as β_n (n = 1, 2, 3, ...), and the solution of $R(r)$ in Equation 9.40 takes the form

$$R(r) = A_n J_0(\beta_n r) \quad \text{with} \quad n = 1, 2, 3, \ldots$$

We recognize that Equation 9.42 is the "characteristic equation" of the differential equation in Equation 9.40 with its roots β_n with n = 1, 2, 3, ... being the eigenvalues as described in Section 4.8.

The complete solution of $R(r)$ including all possible solutions becomes

$$R(r) = \sum_{n=1}^{\infty} A_n J_0(\beta_n r) \tag{9.43}$$

The solution $u(r,t)$ in Equation 9.36 can thus be obtained by the relationship in Equation 9.37 after substitution of $\tau(t)$ in Equation (h) and $R(r)$ in Equation 9.43, resulting in

$$u(r,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha\beta_n^2 t} J_0(\beta_n r) \tag{9.44}$$

We have "lumped" coefficients c_n and A_n in the respective expressions in Equation (h) and Equation 9.43 into another multi-valued constant b_n in Equation 9.44.

The determination of the constant b_n in Equation 9.44 requires the use of the initial condition specified in Equation (d), which leads to

$$\begin{aligned} u(r,0) &= f(r) - T_f \\ &= \sum_{n=1}^{\infty} b_n J_0(\beta_n r) \\ &= b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + \cdots \end{aligned} \tag{9.45}$$

We will multiply both sides of Equation 9.45 by the following summation of Bessel functions:

$$[rJ_0(\beta_1 r) + rJ_0(\beta_2 r) + rJ_0(\beta_3 r) + \cdots]$$

leading to the following expression:

$$\begin{aligned} &[rJ_0(\beta_1 r) + rJ_0(\beta_2 r) + rJ_0(\beta_3 r) + \cdots][f(r) - T_f] \\ &\quad = [rJ_0(\beta_1 r) + rJ_0(\beta_2 r) + rJ_0(\beta_3 r) + \cdots][b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + \cdots] \end{aligned}$$

We may express the above relation in the following form:

$$rJ_0(\beta_n r)[f(r) - T_f] = rJ_0(\beta_n r)[b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + \cdots] \quad \textbf{k}$$

We will integrate both sides of the expression in Equation (k) with respect to variable r:

$$\begin{aligned}&\int_0^a rJ_0(\beta_n r)[f(r) - T_f]dr \\ &= \int_0^a rJ_0(\beta_n r)[b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + \cdots]dr \qquad \text{for } n = 1, 2, 3, \ldots \\ &= \int_0^a rb_n [J_0(\beta_n r)]^2\, dr + \int_0^a rb_1 [J_0(\beta_1 r) J_0(\beta_2 r) + J_0(\beta_1 r) J_0(\beta_3 r) + \cdots]\, dr + \cdots\end{aligned} \quad \textbf{l}$$

According to Fourier–Bessel expansions and integrals (Gray and Mathews, 1966):

$$\int_0^a rJ_0(\beta_m r) J_0(\beta_n r)\, dr = \begin{cases} 0 & \text{if } \beta_m \neq \beta_n \\ \int_0^a r[J_0(\beta_n r)]^2\ dr & \text{if } \beta_m = \beta_n \end{cases}$$

Thus, we have from the above expression that only the first integral on the right-hand side of the above expression has nonzero value. Consequently, we have the expression after integration of the above expression:

$$\int_0^a rJ_0(\beta_n r)[f(r) - T_f]\, dr = b_n \int_0^a r[J_0(\beta_n r)]^2 dr \quad \textbf{m}$$

But we find that integration of the expression in Equation (m) will result in

$$\begin{aligned}\int_0^a rJ_0(\beta_n r)[f(r) - T_f]\, dr &= b_n \int_0^a r[J_0(\beta_n r)]^2 dr \\ &= b_n \frac{a^2}{2}[J_0^2(\beta_n a) + J_1^2(\beta_n a)]\end{aligned} \quad \textbf{n}$$

Since we have already established the relation that $J_0(\beta_n a) = 0$ in Equation 9.42, we will have

$$\begin{aligned}\int_0^a rJ_0(\beta_n r)[f(r) - T_f]\, dr &= b_n \frac{a^2}{2}[J_0^2(\beta_n a) + J_1^2(\beta_n a)] \\ &= b_n \frac{a^2}{2} J_1^2(\beta_n a)\end{aligned}$$

which means that the coefficients b_n in Equation 9.44 are

$$b_n = \frac{2}{a^2 J_1^2(\beta_n a)} \int_0^a r[f(r) - T_f] J_0(\beta_n r)\, dr \quad \textbf{9.46}$$

We may thus obtain the solution of Equation 9.35 via the relation in Equation 9.37 to have the form:

$$T(r,t) = T_f + \sum_{n=1}^{\infty} b_n e^{-\alpha \beta_n^2 t} J_0(\beta_n r) \quad \textbf{9.47}$$

in which the coefficient b_n with n = 1, 2, 3, ... may be computed from Equation 9.46. As in the case in Section 9.5.1, the physical situation in the present case is that the ultimate temperature of the cylinder becomes equal to the surrounding fluid temperature T_f at time $t \to \infty$.

We observe from this case that Bessel functions often appear in the solution of partial differential equations in the cylindrical polar coordinate system.

9.6 Solution of Partial Differential Equations for Steady-state Heat Conduction Analysis

Often, we are required to find the temperature distributions in solids with stable heat flow patterns, which make the temperature distributions in solids independent of time variation — that is, steady-state heat conduction. Mathematical representation of this situation is available by use of the partial differential equation without the term related to time variable *t*. Consequently, the partial differential equation in Equation 9.15 is reduced to the following form:

$$\nabla^2 T(\mathbf{r}) + \frac{Q(\mathbf{r})}{k} = 0 \tag{9.48}$$

where the position vector **r** represents (x,y,z) in the rectangular coordinate system, or (r,θ,z) in the cylindrical polar coordinate system. Equation 9.48 is further reduced to the "Laplace equation" in the following form if no heat is generated by the solid:

$$\nabla^2 T(\mathbf{r}) = 0 \tag{9.49}$$

9.6.1 Steady-state Heat Conduction Analysis in the Rectangular Coordinate System

We will demonstrate the use of the Laplace equation in Equation 9.49 for the temperature distribution in a square plate with the temperature at its three edges maintained constant at 0°C as illustrated in Figure 9.10.

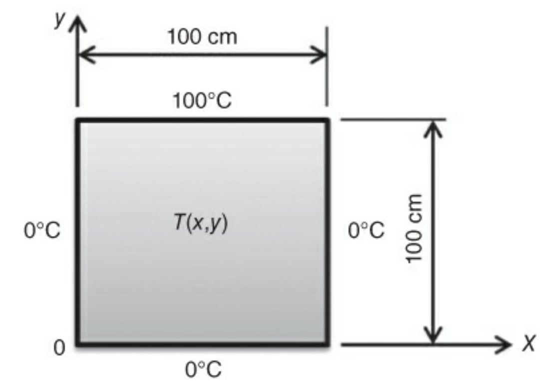


Figure 9.10 Temperature distribution in a plate.

We recognize the situation in which the temperature distribution in the plate is stable with both its flat faces thermally insulated and heat flows in the *x-y* plane. The induced temperature distribution in the plate is expressed by the function $T(x,y)$ as shown in Figure 9.10. Being a steady-state heat flow, the analysis fits the representation of the physical situation by Equation 9.49. We may thus expand this equation to the following form in the *x-y* plane:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0 \tag{9.50}$$

with $0 \le x \le 100$ and $0 \le y \le 100$.

The following boundary conditions apply in the present case:

$$T(x,y)|_{x=0} = T(0,y) = 0 \tag{a1}$$

$$T(x,y)|_{x=100} = T(100,y) = 0 \tag{a2}$$

$$T(x,y)|_{y=0} = T(x,0) = 0 \quad \textbf{a3}$$

$$T(x,y)|_{y=100} = T(x,100) = 100 \quad \textbf{a4}$$

Because both of the variables x and y are within finite bounds, none of the two integral transform methods can be used for the solution of Equation 9.50; we will therefore use the separation of variable technique for the solution. Consequently, we express the solution $T(x,y)$ in Equation 9.50 in the form

$$T(x,y) = X(x)\,Y(y) \quad \textbf{b}$$

in which the function $X(x)$ involves only variable x, and the function $Y(y)$ involves variable y only.

Substituting the relationship in Equation (b) into Equation 9.50 and rearranging terms yields the following:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$$

We will use the same argument as we did in Sections 9.5.1 and 9.5.2 that the only way the above equality can exist is to have both sides of the equality to be equal to the same constant. We will thus have the following equality:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\beta^2 \quad \textbf{c}$$

in which β is the separation constant.

We may derive two ordinary differential equations from Equation (c), one by equating the left-hand side of Equation (c) with the separation constant, and another by equating the right-hand side of the expression with the same separation constant β:

$$\frac{d^2X(x)}{dx^2} + \beta^2 X(x) = 0 \quad \textbf{d}$$

and

$$\frac{d^2Y(y)}{dy^2} - \beta^2 Y(y) = 0 \quad \textbf{e}$$

The relationship in Equation (b) also leads to the following boundary conditions for Equations (d) and (e):

$$X(0) = 0 \quad \textbf{f1}$$

$$X(100) = 0 \quad \textbf{f2}$$

$$Y(0) = 0 \quad \textbf{f3}$$

Equation (d) corresponds to the typical second-order linear differential equation with constant coefficients as presented in Section 8.2. It has a solution in the form

$$X(x) = A\cos\beta x + B\sin\beta x \quad \textbf{g}$$

Applying the first condition in Equation (f1) to Equation (g) will result in $A = 0$ and the second condition in Equation (f2) will lead to the expression:

$$X(100) = B\sin(100\beta) = 0 \quad \textbf{h}$$

The equality in Equation (h) offers the possibilities of having either the coefficient $B = 0$ or $\sin(100\beta) = 0$. Because the coefficient A in Equation (g) is already zero, we cannot allow constant $B = 0$. We thus have sin(100β) = 0, which leads to 100β = $n\pi$ with n = 1, 2, 3, ..., or $\beta = n\pi/100$ with n = 1, 2, 3, ... We may thus express the separation constant β in a multi-valued form as

$$\beta_n = \frac{n\pi}{100} \qquad \text{with } n = 1, 2, 3, \ldots$$

As in the similar situations in the previous two cases in Section 9.5, since sin(100β) = 0 in Equation (h) is the "characteristic equation" for the differential equation in Equation (d), the solutions $\beta = \beta_n$ with n = 1, 2 , 3, ... are eigenvalues as defined in Section 4.8. We thus have the solution of $X(x)$ in the form

$$X(x) = B_n \sin \beta_n x \tag{j}$$

with the constant coefficients B_n and the eigenvalues β_n where n = 1, 2, 3,....

The solution $Y(y)$ in Equation (e) can also be obtained using the method presented in Section 8.2, but we will adopt using the hyperbolic sine and cosine functions instead of exponential functions in the solution. The expression for function $Y(y)$ is

$$Y(y) = C \cosh \beta y + D \sinh \beta y \tag{k}$$

where C and D are arbitrary constants.

The boundary condition in Equation (f3) leads to the coefficient C = 0; thus the form of the function $Y(y)$ will be

$$Y(y) = D \sinh \beta y$$

Because the separation constant β in the above expression is multi-valued, we may express the solution of $Y(y)$ in the following form:

$$Y(y) = D_n \sinh \beta_n y \tag{m}$$

We have obtain the solution $T(x,y)$ of Equation 9.50 after substituting the expressions for $X(x)$ in Equation (j) and $Y(y)$ in Equation (m) into Equation (b), which results in

$$\begin{aligned} T(x, y) &= \sum_{n=1}^{\infty} X(x) Y(y) \\ &= \sum_{n=1}^{\infty} B_n D_n \left(\sin \frac{n\pi}{100} x\right) \left(\sinh \frac{n\pi}{100} y\right) \\ &= \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi}{100} x\right) \left(\sinh \frac{n\pi}{100} y\right) \end{aligned} \tag{n}$$

The unknown coefficients b_n in Equation (n) may be determined using the remaining boundary condition in Equation (a4) that $T(x$,100) = 100, which leads to

$$T(x, 100) = 100 = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi}{100} x\right) (\sinh n\pi)$$

or

$$100 = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin \frac{n\pi}{100} x \tag{p}$$

By following a similar procedure to determination of the coefficient b_n in Equation 9.31 and making use of the orthogonality properties of the harmonic sine and cosine functions in Equation 9.32, we find the expression for the coefficients b_n in the present case to be

$$b_n \sinh n\pi = \frac{2}{100}\int_0^{100} 100 \sin\frac{n\pi}{100}x\,dx = -\frac{200}{n\pi}(\cos n\pi - 1)$$

We thus have

$$b_n = -\frac{200(\cos n\pi - 1)}{n\pi \sinh n\pi} \qquad \text{with } n = 1, 2, 3, \ldots \tag{q}$$

The solution of the temperature distribution in the flat plate in Equation 9.50 with the coefficients b_n expressed in Equation (q) thus has the form

$$T(x,y) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n \sinh n\pi}\sin\frac{n\pi}{100}x \sinh\frac{n\pi}{100}y \tag{9.51}$$

One may substitute cos $n\pi = (-1)^n$ with n = 1, 2, 3, ... in Equation 9.51 and compute the temperature at the geometric center of the plate as T(50,50) = 25.2°C.

9.6.2 Steady-state Heat Conduction Analysis in the Cylindrical Polar Coordinate System

We will demonstrate the use of separation of variables technique for the solution of steady-state heat conduction in a solid in a cylindrical polar coordinate system. This coordinate system is used for engineering analysis involving solids of circular geometry such as disks and cylinders. As demonstrated in Section 9.5.2, in analyses of solids of this geometry, Bessel functions often appear in the solution.

The case we have here involves a solid cylinder of radius a and length L, with temperature at the circumference and the bottom end maintained at 0°C and the temperature at the top surface subjected to a temperature distribution following a specified function $T(r)$ as shown in Figure 9.11.

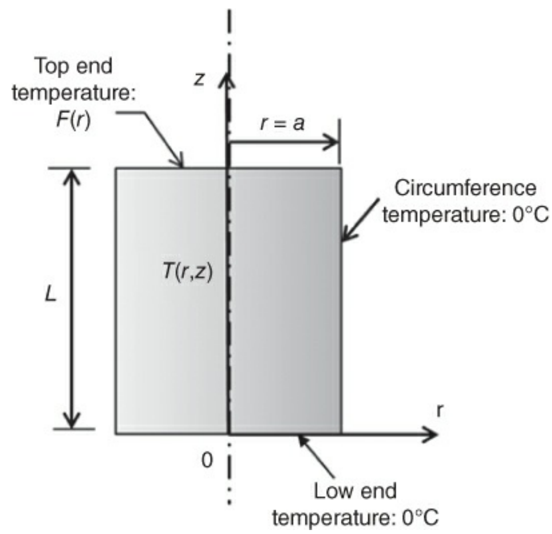


Figure 9.11 Steady-state heat conduction in a solid cylinder.

We recognize the physical situation in which heat flows from the top end of the cylinder in both the radial and longitudinal directions. We may thus represent the temperature in the cylinder by $T(r,z)$ in a cylindrical polar coordinate system as illustrated in Figure 9.11.

The governing partial differential equation for $T(r,z)$ in steady-state heat conduction as described above may be obtained by selecting the appropriate terms in Equation 9.14b in a cylindrical polar coordinate system in the following form:

$$\frac{\partial^2 T(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,z)}{\partial r} + \frac{\partial^2 T(r,z)}{\partial z^2} = 0 \quad \textbf{9.52}$$

with specified boundary conditions:

$$T(a,z) = 0 \quad \textbf{a1}$$

$$T(0,z) \neq \infty \quad \textbf{a2}$$

$$T(r,0) = 0 \quad \textbf{a3}$$

$$T(r,L) = F(r) \quad \textbf{a4}$$

Because both the variables associated with the temperature distribution $T(r,z)$ are valid for finite ranges, we cannot use either of the integral transform methods presented in Section 9.3; we will use the separation of variables technique for the solution of Equation 9.52 by letting

$$T(r,z) = R(r)Z(z) \quad \textbf{b}$$

We obtain the following expression after substituting the assumed relation in Equation (b) into Equation 9.52, followed by a rearrangement of terms:

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)}\frac{\partial R(r)}{\partial r} = -\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} \quad \textbf{c}$$

We note that the left-hand side of Equation (c) contains function $R(r)$ with variable r only, whereas the right-hand side contains a function $Z(z)$ with variable z only. The only way this equality can be valid is making the functions on each side of the above equation equal to the same constant. Consequently, we have the following expression in Equation (d) after the separation of variables r and z:

$$\frac{1}{R(r)}\frac{d^2 R(r)}{dr^2} + \frac{1}{rR(r)}\frac{dR(r)}{dr} = -\frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = -\beta^2 \quad \textbf{d}$$

where β is the separation constant.

Note that we have converted the partial differentiations in Equation 9.52 with respect to both independent variables r and z into two ordinary differential equations in Equation (d).

The constant β appearing in Equation (d) is referred to as the separation constant for Equation 9.52. We will derive two ordinary differential equations by equating the left-hand side of the expression in Equation (d) with $-\beta^2$, and another ordinary differential equation by equating the right-hand side of the expression in Equation (d) with $-\beta^2$. These two ordinary differential equations are shown below:

$$r\frac{d^2 R(r)}{dr^2} + \frac{dR(r)}{dr} + r\beta^2 R(r) = 0 \quad \textbf{e}$$

and

$$\frac{d^2 Z(z)}{dz^2} - \beta^2 Z(z) = 0 \quad \textbf{f}$$

We may express the specified boundary conditions for Equation 9.52 in Equations (a1), (a2), and (a3) as the following equivalent conditions using the relationship shown in Equation (b):

$$R(a) = 0 \quad \textbf{g1}$$

$$R(0) \neq \infty \quad \textbf{g2}$$

$$Z(0) = 0 \quad \textbf{g3}$$

The solution of Equation (e) involves Bessel functions as in the case in Section 9.5.2 and in Equation 9.41:

$$R(r) = AJ_0(\beta r) + BY_0(\beta r) \tag{9.41}$$

The condition specified in Equation (g2) results in having the constant coefficient B = 0, because the second term in the above solution in Equation 9.41 cannot be allowed in the expression because $Y_0(0) \to -\infty$, which leads to an unrealistic physical situation for $T(r,z)$. Consequently, the solution $R(r)$ of Equation (e) has the form

$$R(r) = AJ_0(\beta r)$$

The subsequent application of the condition in Equation (g1) to the above relation results in $R(a) = 0 = AJ_0(\beta a)$, in which we may either have A = 0 or $J_0(\beta a)$ = 0. It is clear that we cannot let the constant A = 0 because the other constant B in the solution of $R(r)$ in Equation 9.41 already equals zero. Letting A = 0 will make $R(r)$ = 0 at all times, which will lead to a trivial solution of $T(r,z)$. We thus take the equation for the separation constant β to be the solutions of $J_0(\beta a)$ = 0. This equation is the characteristic equation of Equation (e) and it will lead to multiple values of the separation constant β, as the equation $J_0(x)$ = 0 has in theory an infinite number of roots x_1, x_2, x_3, ..., as demonstrated in Example 9.5. We may thus express the separation constant β in the form β_n with n = 1, 2, 3, ..., and the solution for the function $R(r)$ should include all possible values of β_n

$$R(r) = A_n J_0(\beta_n r) \tag{h}$$

in which A_n are the constant coefficients corresponding to the values of β_n obtained from the following equation:

$$J_0(\beta_n a) = 0 \tag{j}$$

with n = 1, 2, 3, ... The solution of Equation (f) can be expressed as

$$Z(z) = C\cosh(\beta z) + D\sinh(\beta z) \tag{k}$$

Substituting the condition $Z(0)$ = 0 for Equation (a3) into Equation (k) leads to the constant C = 0. We will thus have

$$Z(z) = D\sinh(\beta z)$$

Since the solution $Z(z)$ in the above expression also involves the separation constants β_n with n = 1, 2, 3, ..., we need to express $Z(z)$ accordingly with β_n in the form

$$Z(z) = D_n \sinh(\beta_n z) \tag{m}$$

We can thus express the solution $T(r,z)$ in Equation 9.52 in the following form with the solutions of $R(r)$ in Equation (h) and $Z(z)$ in Equation (m):

$$T(r,z) = [A_n J_0(\beta_n r)][D_n \sinh(\beta_n z)] \quad \text{with} \quad n = 1, 2, 3, \ldots$$

We need to include all valid solutions corresponding to the values of β_n and express the complete solution of $T(r,z)$ in the following form:

$$T(r,z) = \sum_{n=1}^{\infty} b_n[J_0(\beta_n r)](\sinh \beta_n z) \tag{9.53}$$

We note that the multi-valued constant coefficient b_n appearing in Equation 9.53 replaces the product of two other multi-valued constants A_n and D_n as in the analyses in preceding cases.

The constant coefficients b_n in Equation 9.53 may be determined by using the Fourier–Bessel series as we did in Section 9.5.2 with the use of the remaining boundary condition in Equation (a4). We will have the following expression:

$$b_n \sinh \beta_n L = \frac{2F(r)}{L^2 J_1^2(\beta_n L)} \int_0^L r J_0(\beta_n r)\, dr$$

from which we obtain the expression for b_n:

$$b_n = \frac{2F(r)}{L^2 J_1^2(\beta_n L) \sinh \beta_n L} \int_0^L r J_0(\beta_n r)\, dr \quad \text{with} \;\; n = 1, 2, 3, \ldots \qquad \textbf{n}$$

We thus have the solution of temperature distribution in the solid cylinder $T(r,z)$ in Equation 9.53 with the coefficient b_n expressed in Equation (n) with specified temperature variation $F(r)$ at the top edge, and the separation constants β_n being the roots of Equation (j).

9.7 Partial Differential Equations for Transverse Vibration of Cable Structures

Engineers often are required to carry out vibration analysis of long flexible cable structures subjected to external forces. Cable structures are common in places such as power transmission lines shown in Figure 9.12, guy wire supports in Figure 9.13, and suspension bridges in Figure 9.14.

Figure 9.12 Long cables in electric power transmission structures.

Figure 9.13 Guy wire support for a tall radio transmission tower.

Figure 9.14 Golden Gate suspension bridge.

These cable structures, flexible in nature, are vulnerable to resonant vibrations as described in Chapter 8. The galloping motions that often develop in resonant vibration of these structures can result in devastating structural failure. Rupture of long power transmission cables resulting from such vibration often occurs in places with cold climate in which freezing rain can build up heavy icicles on the cables overnight. The significant weight gain (and thus an increase of mass) of the cable causes the cable to vibrate violently with gale-force winds, and in many cases can result in breaking of the cables. Many transmission towers have also been destroyed due to violent vibration of the transmission cable lines, resulting in millions of dollars of property loss to the power companies in these regions. This is thus an important subject for mechanical and structural engineers in the safe design of structures of this type.

9.7.1 Derivation of Partial Differential Equations for Free Vibration of Cable Structures

Mathematical modeling of vibration analysis of cable structures begins with the illustration of the cable movements as in Figure 9.15, in which a cable of length L is at a static equilibrium condition with both its ends fixed at supports and with an initial sag due to its own weight and the associated tension in the cable. This initial shape of the free-hung cable is represented by function $f(x)$ as illustrated in the figure.

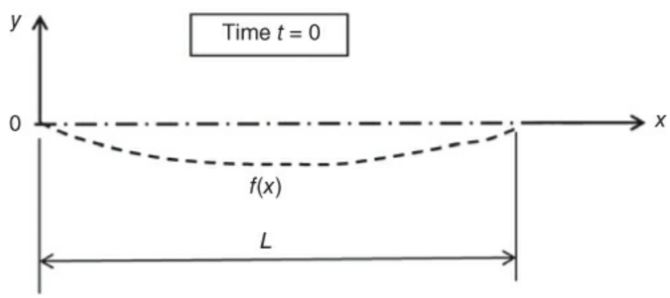


Figure 9.15 A long cable in a static equilibrium state.

Formulation of the differential equation for vibration of long cables is based on the law of conservation of momentum, and in particular, the use of Newton's second law in dynamics.

Like all engineering analyses, the mathematical modeling of vibrating flexible cable structures begins with a number of assumptions and idealizations such as were described in Stage 2 in Section 1.4. Some of these assumptions are presented below.

1. The cable is as flexible as a string. This means that the cable has no strength to resist a bending moment and there is no shear force associated with its deflection. The mathematical model that we will develop subsequently will be more relevant to the case with long cables for their inherent flexibility.

2. There exists a tension in the string in its free-hung static state as shown in Figure 9.15. This tension is so large that the weight, but not the mass, of the cable is neglected in the analysis.

3. Each small segment of the cable along its length, that is, the segment with length Δx, moves only in the vertical direction during vibration.

4. The vertical movement of the cable along the length is small, so the slope of the deflection curve is small.

5. The mass of the cable along the length is constant, that is, the cable is made of the same material along its length.

Figure 9.16 illustrates the vibration of the cable produced by a small instantaneous disturbance in the vertical direction (or y-direction)—a situation similar to the free vibration that we dealt with for mass–spring systems in Chapter 8.

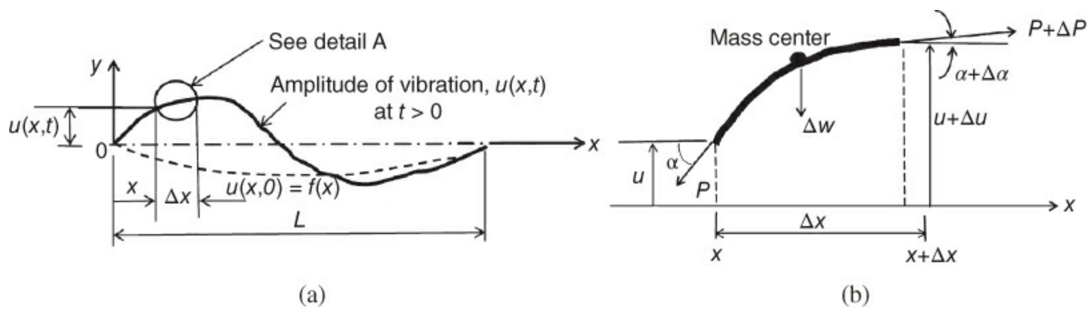


Figure 9.16 A vibrating long cable. (a) Shape of the vibrating cable at time t. (b) Forces on the cable in Detail A.

Let the mass per unit length of the string be designated by m. The total mass of string in an incremental length Δx will thus be ($m\,\Delta x$). The condition for a dynamic equilibrium according to Newton's second law presented in the *equation of motion* obeys the relationship

$$\underset{\sum F}{\text{Total applied forces}} = \underset{m}{\text{Mass}} \times \underset{a}{\text{Acceleration}}$$

Let us consider the forces acting on the small segment of the string shown in Detail A of Figure 9.16b in the free-body diagram in Figure 9.17. From Figure 9.17, we may express the equation of motion of this string segment in the y-direction as

$$\sum F_y = -(P + \Delta P)\sin(\alpha + \Delta\alpha) - P\sin\alpha + (m\,\Delta x)\frac{\partial^2}{\partial t^2}\left(u + \frac{\Delta u}{2}\right) = 0$$

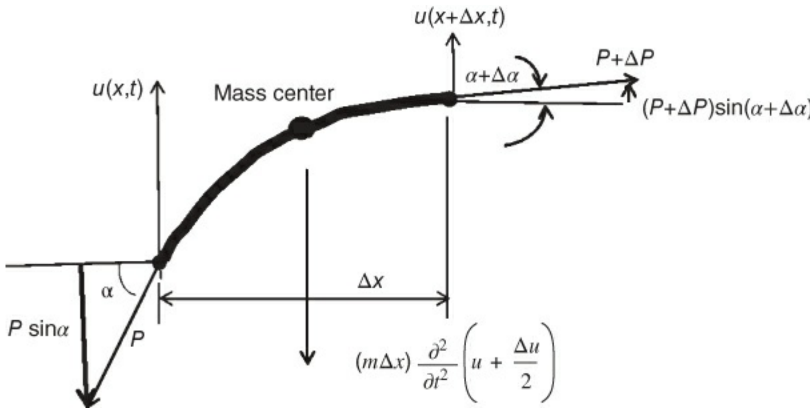


Figure 9.17 Dynamic forces acting on a cable segment.

We note that ΔP is small, so that $\Delta P\sin(\alpha+\Delta\alpha)\rightarrow 0$; we thus have

$$-P\sin(\alpha+\Delta\alpha)-P\sin\alpha+(m\,\Delta x)\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)=0$$

For the small angle α and thus very small $\Delta\alpha$, the following relationship exists:

$$\sin(\alpha+\Delta\alpha)\approx\tan(\alpha+\Delta\alpha)=\frac{\partial u(x+\Delta x,t)}{\partial x}$$

holds and

$$\sin\alpha\approx\tan\alpha=\frac{\partial u(x,t)}{\partial x}$$

the above equation of motion can be expressed as

$$P\left[\frac{\partial u(x+\Delta x,t)}{\partial x}-\frac{\partial u(x,t)}{\partial x}\right]=(m\,\Delta x)\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)$$

or, after dividing both sides by Δx, as

$$P\frac{\dfrac{\partial u(x+\Delta x,t)}{\partial x}-\dfrac{\partial u(x,t)}{\partial x}}{\Delta x}=m\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)$$

We recall the definition of the derivative of a function $y(x)$ in Section 2.2.5 as

$$\frac{dy(x)}{dx}=\lim_{\Delta x\to\infty}\frac{y(x+\Delta x)-y(x)}{\Delta x}$$

We may use this definition to arrive at the following relation for partial derivatives:

$$\lim_{\Delta x\to\infty}\frac{\dfrac{\partial u(x+\Delta x,t)}{\partial x}-\dfrac{\partial u(x,t)}{\partial x}}{\Delta x}=\frac{\partial^2 u(x,t)}{\partial x^2}$$

The equation of motion of the string can be further simplified to take the form

$$P\frac{\partial^2 u(x,t)}{\partial x^2}=m\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)$$

But since we have assumed that Δu is small, and $\Delta u/2$ is also small enough to be neglected, we thus have the equation for the instantaneous deflection of the string at x as

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \textbf{9.54}$$

in which $a = \sqrt{P/m}$ with P = tension in the string, with the unit of newton (N), and m = mass of the cable per unit length, with the unit of kg/m. The unit for the constant a in Equation 9.54 is thus m/s. This unit is derived from the relations that the tension P has the unit of newton, or kg-m/s^2.

Equation 9.54 is often referred to as the "wave equation" that describes wave motion, with $u(x,t)$ being the amplitude of the wave and the constant coefficient a being the wave propagation speed.

The solution $u(x,t)$ in Equation 9.54 for vibration of strings (or long flexible cables) represents the instantaneous deflection (or amplitude of vibration) of the cable in a free vibration state. Some common specified conditions used in solving this equation are as follows.

The initial condition at time t = 0:

$$u(x,t)|_{t=0} = u(x,0) = f(x) \quad \textbf{9.55a}$$

and the initial state at static equilibrium with

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \dot{u}(x,0) = 0 \quad \textbf{9.55b}$$

Also the following end (boundary) conditions with fixed ends:

$$u(x,t)|_{x=0} = u(0,t) = 0 \quad \textbf{9.56a}$$

and

$$u(x,t)|_{x=L} = u(L,t) = 0 \quad \textbf{9.56b}$$

9.7.2 Solution of Partial Differential Equation for Free Vibration of Cable Structures

We will use the separation of variables technique to solve Equation 9.54 for free vibration of a long flexible cable with the initial and end conditions specified in Equations 9.55 and 9.56.

We recognize that the cable is deformed by its own weight into a shape that can be described by a function $f(x)$ at time t = 0, and the vibration is induced to the cable by a small disturbance that causes the cable to vibrate in the direction perpendicular to the x-axis as shown in Figure 9.18. We will determine the amplitude of vibration of the cable $u(x,t)$ at time $t = 0^+$, that is, having the cable released from its initially deformed state $f(x)$. The instantaneous position of the cable, or the amplitude of vibration of the cable, is represented by a function $u(x,t)$. Physically this function $u(x,t)$ is the vertical displacement of any given point of the cable at distance x from the left end at time t = 0, as illustrated in Figure 9.18.

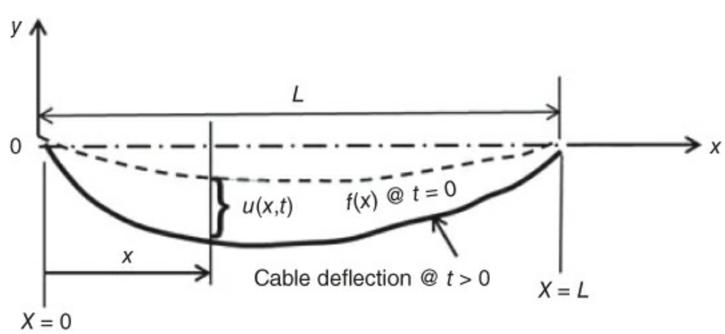


Figure 9.18 A long cable subjected to lateral vibration.

The situation in this case corresponds to the description of free vibration of a flexible cable, which makes Equation 9.54 relevant for the current problem. We thus have the partial differential equation for the problem in the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{9.54}$$

We may use the Laplace transform method by transforming the variable t to the Laplace transform domain because the variable t covers the range $(0,\infty)$, which legitimizes the use of this method. We may also use the separation of variables technique for the solution of Equation 9.54. We will choose the latter technique for the solution by letting

$$u(x,t) = X(x)\,T(t) \tag{9.57}$$

in which the function $X(x)$ involves variable x only in Equation 9.57, and the other function $T(t)$ involves variable t only.

Substituting the expression in Equation 9.57 into Equation 9.54, one gets the following expression after rearranging the terms:

$$\frac{1}{a^2 T(t)} \frac{d^2 T(t)}{dt^2} = \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}$$

Inspection of the above relationship will reveal that the left-hand side (LHS) of the above expression is a function of variable t only, whereas the right-hand side (RHS) is a function of variable x only. The only possibility of having such a relationship to be valid is that both sides of the above equation are equal to the same constant (LHS = RHS = constant), or in mathematical form:

$$\frac{1}{a^2 T(t)} \frac{d^2 T(t)}{dt^2} = \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -\beta^2 \tag{9.58}$$

in which β is the "separation constant." A negative sign is attached to the square of β to ensure that both the LHS and RHS of Equation 9.58 are equal to a *negative* constant. The reason for this is that a positive constant will lead to an "unbounded" solution of Equation 9.54.

The relationship in Equation 9.58 is a critical step in the solution, as it allows us to obtain two separate ordinary differential equations, one for the function $X(x)$ and the other for the function $T(t)$. Thus, letting LHS of Equation 9.58 = $-\beta^2$, we will have the following differential equation for $T(t)$:

$$\frac{d^2 T(t)}{dt^2} + a^2 \beta^2 T(t) = 0 \tag{9.59}$$

The conditions associated with Equation 9.59 can be derived by substituting the expression in Equations 9.55a and 9.55b into Equation 9.57 to give the following forms for the initial conditions:

$$u(x,t)|_{t=0} = u(x,0) = X(x)T(0) = f(x) \tag{9.60a}$$

$$\left.\frac{dT(t)}{dt}\right|_{t=0} = 0 \tag{9.60b}$$

If we let the term on the right-hand side of Equation 9.58 also be equal to $-\beta^2$, we will have the other differential equation for $X(x)$:

$$\frac{d^2 X(x)}{dx^2} + \beta^2 X(x) = 0 \tag{9.61}$$

Again, the conditions associated with Equation 9.61 can be derived from Equations 9.56a and 9.56 using Equation 9.57, resulting in

$$X(0) = 0 \quad \textbf{9.62a}$$

$$X(L) = 0 \quad \textbf{9.62b}$$

One realizes that both differential equations in Equations 9.59 and 9.61 are homogeneous second-order ordinary differential equations with constant coefficients. The general solutions of this type of differential equation may be obtained using the method described in Section 8.2.

We thus have the solution of Equation 9.59 as

$$T(t) = A\sin(\beta at) + B\cos(\beta at) \quad \textbf{9.63}$$

and the solution of Equation 9.61 as

$$X(x) = C\sin(\beta x) + D\cos(\beta x) \quad \textbf{9.64}$$

in which A, B, C, and D are arbitrary constants to be determined by the conditions stipulated in Equations b and 9.62a,b.

By substituting the expressions in Equations 9.63 and 9.64 into Equation 9.57, we will have the solution $u(x,t)$ of the problem:

$$u(x,t) = [A\sin(\beta at) + B\cos(\beta at)][C\sin(\beta x) + D\cos(\beta x)]$$

We are now ready to determine the arbitrary constants A, B, C, and D in the above solution. Let us first use the condition in Equation 9.62a: $X(0) = 0$ leads to

$$C\sin(\beta \times 0) + D\cos(\beta \times 0) = 0$$

from which we have $D = 0$. We thus have $X(x) = C\sin(\beta x)$. Now, if we use the other condition in Equation 9.62b, we will have

$$X(L) = 0 = C\sin(\beta L)$$

At this point, we have the choice of letting $C = 0$ or $\sin(\beta L) = 0$ from the above relationship. A close look at these choices will show that $C \neq 0$ if we want to avoid a trivial solution of $u(x,t)$. We thus have

$$\sin(\beta L) = 0 \quad \textbf{9.65}$$

Equation 9.65, like those in the previous cases in Sections 9.5 and 9.6, is a characteristic equation of the differential equation in Equation 9.61; the infinite number of valid solutions with $\beta = \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots, n\pi$, in which n is an integer, are the eigenvalues as described in Section 4.8. We may thus obtain the values of the separation constant, β as

$$\beta_n = \frac{n\pi}{L} \quad (n = 0, 1,2, 3,4 \ldots) \quad \textbf{9.66}$$

The solution of the partial differential equation in Equation 9.54 thus becomes

$$u(x,t) = \left(A_n \sin\frac{n\pi}{L}at + B_n\cos\frac{n\pi}{L}at\right) C_n \sin\frac{n\pi}{L}x$$

By combining the constants, A_n, B_n, and C_n and for $n = 1, 2, 3, \ldots$ one may express the above expression in the following form with $a_n = A_nC_n$ and $b_n = B_nC_n$:

$$u(x,t) = \left(a_n \sin\frac{n\pi}{L}at + b_n\cos\frac{n\pi}{L}at\right) \sin\frac{n\pi}{L}x$$

with $n = 1, 2, 3, \ldots$, and a_n and b_n being the new arbitrary constants.

We may determine the constants, a_n and b_n, by using the condition given in Equation 9.55b in the above expression:

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0 = \frac{n\pi a}{L}\left.\left(a_n \cos\frac{n\pi at}{L} - b_n \sin\frac{n\pi at}{L}\right)\right|_{t=0} \sin\frac{n\pi}{L}x$$

leading to $a_n\left(\frac{n\pi a}{L}\right)\sin\frac{n\pi}{L}x = 0$

Since $\sin(n\pi/L)x \neq 0$ in the above expression, we see that the coefficients a_n = 0. We will thus have

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi a}{L} t \sin\frac{n\pi a}{L} x$$

The only remaining undetermined multi-valued constant coefficients b_n in the above expression can be determined using the remaining unused condition in Equation 9.55a; that is, $u(x,0) = f(x)$, or

$$u(x,t)|_{t=0} = u(x,0) = f(x)$$

We thus have the following relationship:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \textbf{9.67}$$

with the function $f(x)$ on the LHS in Equation 9.67 being the given initial condition of the partial differential equation of Equation 9.54. The coefficients b_n (n = 1,2,3, ...) in the RHS of the equation will be determined by the procedures presented in Section 9.5.1, as follows:

1. Multiply both sides of Equation 9.67 by a function $\sin(n\pi x/L)$ as follows:

$$\begin{aligned}\sin\frac{n\pi x}{L} f(x) &= \left(\sin\frac{n\pi x}{L}\right)\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\\ &= \sum_{n=1}^{\infty} b_n\left(\sin\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L}\end{aligned}$$

2. Integrate both sides of the above equation:

$$\begin{aligned}\int_0^L \sin\frac{n\pi x}{L} f(x)\,dx &= \int_0^L \sum_{n=1}^{\infty} b_n\left(\sin\frac{n\pi x}{L}\right)\sin\frac{n\beta\pi x}{L}\,dx\\ &= \sum_{n=1}^{\infty}\int_0^L b_n\left(\sin\frac{n\pi x}{L}\right)^2 dx\end{aligned}$$

The orthogonality of integration of sine functions leads to the expression

$$\begin{aligned}\int_0^p \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx &= 0 \quad \text{if } m \neq n,\\ &= \frac{p}{2} \quad \textit{if } m = n\end{aligned}$$

The above expression leads to the coefficients b_n in Equation 9.67 as

$$\int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx = \frac{L}{2}$$

from which we will have

$$\int_o^L \sin\frac{n\pi x}{L} f(x)\,dx = b_n\left(\frac{L}{2}\right) \quad \text{with } n = 1,2,3,4,\ldots$$

We thus establish the expression for the multi-valued constant b_n as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{with } n = 1, 2, 3, 4, 5, \ldots \tag{9.68}$$

With the coefficients b_n determined by Equation 9.68, we may express the complete solution of Equation 9.54 as

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\left(\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right)\cos\frac{n\pi a t}{L}\sin\frac{n\pi x}{L} \tag{9.69}$$

9.7.3 Convergence of Series Solutions

Solutions of partial differential equations by the separation of variables technique such as presented in Sections 9.5 and 9.6 include summations of an infinite number of terms associated with the infinite number of roots of the transcendental equation (or characteristic equations as mentioned in Chapter 4). The solution in Equation 9.69 for the partial differential equation in (9.54) is also in the form of infinite series. Numerical solutions of these equations can be obtained by summing up the solutions with each assigned value of n, that is with n = 1, 2, 3, ... up to a very large integer number. In normal circumstances, The series should converge fairly rapidly, so one needs only to sum up approximately a dozen terms using n = 1, 2, ..., 12. However, the effect of the rate of convergence of infinite series, such as that in Equation 9.69, on the accuracy of the analytical results remains a concern to engineers in their analyses.

We will demonstrate the convergence of the following fictitious series solution that is similar but not identical to that shown in Equation (9.69), with selected solution point at x = 5 and t = 1.

$$u(5,1) = \sum_{n=1}^{\infty} \frac{2}{20}\left[\int_0^{20}\left(0.25\sin\frac{n\pi}{20}x\right)dx\right]\cos\left(\frac{120n\pi}{20}t\right)\sin\frac{n\pi}{20}x$$

or in final form as

$$u(5,1) = \frac{1}{40}\sum_{n=1}^{\infty}\left[\cos(6n\pi)\sin\frac{n\pi}{4}\left(\int_0^{20}\sin\frac{n\pi}{20}x\,dx\right)\right]$$

We will let the numerical solution of u(5,1) be the summation of all terms denoted by n:

$$u(5,1) = u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + n_n$$

where u_1, u_2, u_3, u_4, u_5, ..., u_n are valid solutions of u(5,1) with n = 1, 2, 3, 4, 5, ..., n. Valid solutions with n up to 16 have been computed and their respective values are listed in the following table:

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
1.6	3.8	2.8	0	−9.38	3.22	9.38	0	−5.63	1.14	5.63	0	−4.02	5.77	4.02	0
E−2	E−2	E−2		E−3	E−3	E−3		E−3	E−3	E−3		E−3	E−4	E−3	

The contribution of the individual valid solutions with u_n (n = 1, 2, 3, 4, 5, ..., 30) is plotted in Figure 9.19.

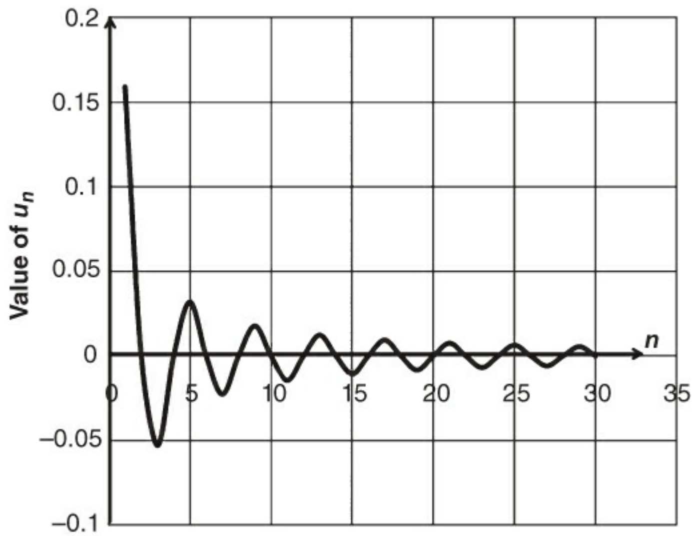


Figure 9.19 Convergence of an infinite series solution.

We envisage from Figure 9.19 that the magnitude of individual terms in the infinite series diminishes rapidly with the increasing *n*-values. This trend indicates that contributions of individual terms in the infinite series solution such as in Equation 9.69 also diminish as the *n*-value increases. Reasonable accuracy of the solution for this particular case is considered to be obtained with *n*-value less than 20 in the numerical solution. It is plausible to assume that reasonable accuracy of the infinite series solution of long cable vibration may be achieved by including fewer than 20 terms in the series solutions.

9.7.4 Modes of Vibration of Cable Structures

We learned from the fundamentals of the mechanical vibration of solids in Section 8.9 that any solid structure made of *elastic material* with a given *geometry* possesses various *modes of vibration*. Each mode offers a unique "deformed shape" of the structure in the vibration. In the case of free vibration of cable structures, the "peaks" and "valleys" in the shape of the structure differ in different modes. The analysis that determines various modes of vibration of a structure is called *modal analysis*. This analysis is of great importance for any structure that is vulnerable to mechanical vibration, and from it engineers may determine how to design cable structures that are likely subjected to intermittent loads, such as wind loads, at certain locations in order to avoid resonant vibration. Additionally, modal analysis will also indicate *where* and *when* the structure may experience maximum amplitude of vibration. Such information could offer the user of the structures the option of not installing delicate attachments at these sensitive locations.

One should bear in mind that modal analysis is also a part of free-vibration analysis of structures with no externally applied load. Elastic material properties and the geometry of the structure are the only required conditions for such analyses.

In the following we will illustrate the first three modes of a vibrating string using the solution given in Equation 9.69 in the form

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{L}\left(\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right)\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L} \tag{9.69}$$

or

$$u(x,t)=\sum_{n=1}^{\infty}b_n\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{9.68}$$

The *mode shapes of the vibrating cable in this case are determined by the assigned distinct n-value in Equation* 9.69 *with the coefficients* b_n *evaluated in Equation* 9.68 *with the same n-value*. Following are the shapes of the vibrating cable in the first three modes with $n = 1$ for mode 1, $n = 2$ for mode 2, and $n = 3$ for mode 3 vibration

The shape of the vibrating cable in **mode 1 with *n* = 1**

We will get the magnitudes of the vibrating cable $u_1(x,t)$ from Equation 9.69 and the coefficient b_1 from Equation 9.68 with $n = 1$ as follows:

$$u_1(x,t) = \left(b_1 \cos\frac{\pi a t}{L}\right)\sin\frac{n\pi}{L}x \text{ and } b_1 = \frac{2}{L}\int_0^L f(x)\sin\frac{\pi x}{L}dx = \text{constant } t$$

The above integral will result in a constant value of the coefficient b_1 with the given function $f(x)$. We may envisage that the shape of the vibrating cable will follow a sine function with its maximum amplitude at any given instant obtained by a $\sin[(\pi/L)x]$ function. Graphically, it can be illustrated in Figure 9.20. We note that the two ends of the string are fixed and have zero amplitude of vibration. The maximum amplitudes of vibration occur at the mid-span of the cable. The corresponding frequency of vibration for this mode of vibration is the coefficient of the argument in the cosine function divided by 2π in the general solution of $u(x,t)$ in Equation 9.69, or as shown in Equation 9.70 in the case of mode 1 vibration.

$$f_1 = \frac{\pi a/L}{2\pi} = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{P}{m}} \tag{9.70}$$

in which P = the tension in the cable (N) and m = the mass of the cable per unit length (kg/m).

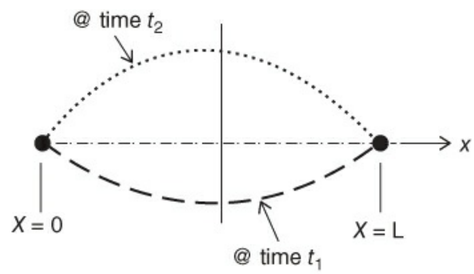


Figure 9.20 Shapes of the cable in mode 1 vibration.

The frequency f_1 in Equation 9.70 has a unit of Hertz (Hz).

The shape of the vibrating cable in mode 2 with *n* = 2 Likewise, the shape of the vibrating cable in mode 2 may be obtained in a similar way by using Equations 9.68 and 9.69 with $n = 2$, as shown in the following expressions:

$$u_2(x,t) = \left(b_2 \cos\frac{2\pi a}{L}t\right)\sin\frac{2\pi}{L}x \tag{9.71a}$$

where

$$b_2 = \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi x}{L}dx = \text{constant } t \tag{9.71b}$$

We recognize from Equation 9.68 that $u_2(t) = 0$ at $x = 0$, $L/2$, and L. Graphically, the amplitudes in Equation 9.71a can be illustrated in Figure 9.21.

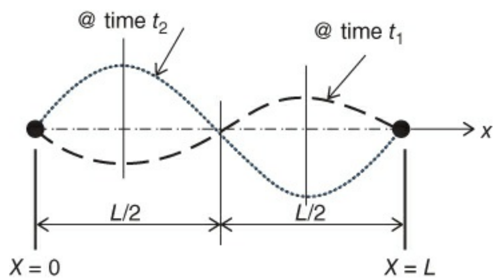


Figure 9.21 Shapes of the cable in mode 2 vibration.

It is interesting to note from Figure 9.21 that there are now three nodes at which the amplitude of vibration is zero. Physically, it means that there are two spots along the cable at which maximum amplitudes of vibration occur at $L/4$ and $3L/4$. The corresponding frequency is

$$f_2 = \frac{2\pi a/L}{2\pi} = \frac{a}{L} = \frac{1}{L}\sqrt{\frac{P}{m}} \tag{9.72}$$

We may find mode 3 of the vibrating cable from Equations 9.68 and 9.69 with $n = 3$ in the following expressions:

$$u_3(x,t) = \left(b_3 \cos\frac{3\pi a}{L}t\right) \sin\frac{3\pi}{L}x \tag{9.73a}$$

where

$$b_3 = \frac{2}{L}\int_0^L f(x) \sin\frac{3\pi x}{L} dx = \text{constant} \tag{9.73b}$$

We note from Equation 9.73a that there are four nodes in this mode of vibration at $x = 0$, $L/3$, $2L/3$, and L as illustrated in Figure 9.22. The corresponding frequency is

$$f_3 = \frac{3\pi a/L}{2\pi} = \frac{3a}{2L} = \frac{3}{2L}\sqrt{\frac{P}{m}} \tag{9.74}$$

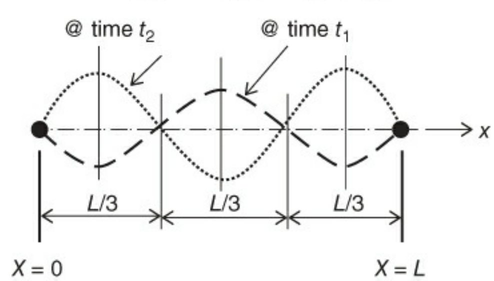


Figure 9.22 Shapes of the cable in mode 3 vibration.

Example 9.6

A flexible cable 10 m long is fixed at both ends with a tension of 500 N in the free-hung state. The cable has a diameter of 1 cm and a mass density of 2.7 g/cm^3. The cable begins to vibrate due to an instantaneous but small disturbance from its initial shape that can be described by the function $f(x) = 0.005x(1-x/10)$, in which x is the coordinate along the length of the cable with x = 0 at one of the two fixed ends. Determine the following:

a. The differential equation for the amplitudes of vibration of the cable represented by $u(x,t)$ in the unit of meters, in which t is the time into the vibration with the unit of seconds.

b. The mathematical expressions for the initial and end conditions.

c. The solution for $u(x,t)$ of the differential equation in the unit of meters.

d. The amplitude of the vibrating cable in mode 1, that is, $u_1(x,t)$ with the magnitude and location of the maximum deflection of the cable in this mode of vibration.

e. The numerical values of the frequencies of the first and second modes of vibration.

f. The physical significance of these mode shapes to the design engineer.

Solution:

The present problem involves the following specific conditions:

The length of the cable L = 10 m and its diameter d = 1 cm = 0.01 m.

The cable is made of aluminum with a mass density ρ = 2.7 g/cm^3.

The cable is subjected to a tension P = 500 N and with initial sag described by the function $f(x)$:

$$f(x) = 0.005x\left(1 - \frac{x}{10}\right)$$

a.

The applicable differential equation for the amplitudes of vibration is Equation 9.54 with the constant coefficient a in that equation to be determined by $a = \sqrt{P/m}$, where P = the tension in the cable = 500 N and m = the mass per unit length, which needs to be computed from the given values. The mass per unit length of the cable is $m = M/L$, where M = the total mass of the cable $M = \rho V$, with V being the volume of the cable. We will get the volume of the cable from the expression $V = [(\pi d^2/4)L] = 7.85 \times 10^{-4}\ \text{m}^3$ with d = the given diameter of the cable. We will thus have the total mass of the cable $M = \rho V = (2.7 \times 10^3)(7.85 \times 10^{-4})$ = 2.12 kg, meaning the mass per unit length of the cable is 0.212 kg/m.

The constant coefficient a in Equation 9.54 according to the expression given above is

$$a = \sqrt{\frac{P}{m}} = \sqrt{\frac{500}{0.212}} = 48.56 \text{ m/s}$$

We will thus need to solve Equation 9.54 with a^2 = 2358.5 on the right-hand side of this equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 2358.5 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{a}$$

b.

The mathematical expressions of the initial and end conditions:

The problem requires satisfying the initial and end conditions expressed mathematically as

The initial conditions are

$$u(x,t)|_{t=0} = u(x,0) = f(x) = 0.005x\left(1 - \frac{x}{10}\right) \tag{b1}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \dot{u}(x,0) = 0 \tag{b2}$$

and the end conditions are

$$u(x,t)|_{x=0} = u(0,t) = 0 \tag{c1}$$

$$u(x,t)|_{x=L} = u(L,t) = u(10,t) = 0 \tag{c2}$$

c.

The solution of $u(x,t)$ of Equation (a) satisfying the conditions in Equations (b1) and (b2) and Equations (c1) and (c2) will be obtained as follows. The solution of Equation (a) is similar to that of Equation 9.69 with a = 48.56 m/s in the following expression:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{10}\left\{\left[\int_0^{10} 0.005x\left(1 - \frac{x}{10}\right)\sin\frac{n\pi x}{10}\,dx\right]\left[\cos\frac{n\pi(48.56)}{10}t\right]\left[\sin\frac{n\pi}{10}x\right]\right\}$$

or

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{5}\left\{\left[\int_0^{10} 0.005x\left(1 - \frac{x}{10}\right)\sin\frac{n\pi x}{10}\,dx\right][\cos 15.25nt][\sin 0.314nx]\right\} \tag{d}$$

Using the integrals available from the handbook (Zwillinger, 2003),

$$\int x\sin\alpha x\,dx = \frac{1}{\alpha^2}\sin\alpha x - \frac{x}{\alpha}\cos\alpha x$$

and

$$\int x^2\sin\alpha x\,dx = \frac{2x}{\alpha^2}\sin\alpha x + \frac{2 - \alpha^2x^2}{\alpha^3}\cos\alpha x$$

we evaluate the integral in Equation (d) in the form

$$\int_0^{10}\left[0.005x\left(1 - \frac{x}{10}\right)\sin\frac{n\pi x}{10}\,dx\right] = -\frac{0.5(-1)^n}{n\pi}\left(1 + \frac{2 - n^2\pi^2}{n^3\pi^3}\right)$$

The solution in Equation (d) thus takes the form

$$u(x,t) = -0.1\sum_{n=1}^{\infty}\frac{1}{n\pi}\left(1 + \frac{2 - n^2\pi^2}{n^3\pi^3}\right)(\cos 15.25nt)(\sin 0.314nx) \tag{e}$$

d.

The amplitude of the vibrating cable in mode 1, that is, $u_1(x,t)$ with the magnitude and location of the maximum deflection of the cable in this mode of vibration. The required solution is obtained by letting n = 1 in Equation (e) as f

$$u_1(x,t) = -0.1\left[\frac{1}{\pi}\left(1+\frac{2-\pi}{\pi^3}\right)\right]\cos 15.25t \sin 0.314x$$
$$= -0.02376 \cos 15.25t \sin 0.314x$$

The maximum amplitude occurs at the mid-span of the cable at x = 5 m, and at the time given by cos 15.25t = 1.0. We thus have the maximum amplitude $u_{1,\max}$ = 0.023 76 m, or 2.376 cm, at x = 5 m and at the time given by 15.25t = π, or time t = π/15.25 = 0.2second.

e. *The numerical values of the frequencies of the first and second mode of vibration:* We may use Equations 9.70 and 9.72 to compute the numerical values of the frequencies of the first and second modes of vibration as follows:

$$f_1 = \frac{1}{2L} = \frac{1}{2\times 10}\sqrt{\frac{500}{0.212}} = 2.43 \text{ Hz} \quad \text{for mode 1}$$

$$f_2 = \frac{1}{L}\sqrt{\frac{P}{m}} = \frac{1}{10}\sqrt{\frac{500}{0.212}} = 4.86 \text{ Hz} \quad \text{for mode 2}$$

f. *The physical significance of these mode shapes to the design engineer:* Engineers will use the outcomes of the above modal analysis to advise the users of this cable structure on the possibility of potentially devastating resonant vibration of the cable structure should the frequency of an applied cyclic force, such as wind force, coincide with any of the natural frequencies, *such as* f_1 = 2.43 Hz, f_2 = 4.86 Hz, ... computed in part (e). The users will also be made aware of the locations where maximum amplitudes of vibration may occur as indicated by the mode shapes in the modal analysis. The users should avoid placing delicate attachments at these locations to avoid potential damage due to excessive vibration there.

9.8 Partial Differential Equations for Transverse Vibration of Membranes

Solids of plane geometry commonly appear in machines and structures. Thin plates can be as small as micrometers in thickness in printed electronic circuit boards or as large as floors in building structures. Like flexible cables, thin flexible plates are normally vulnerable to transverse vibration. In some cases, these plates may rupture due to resonant vibration, resulting in significant loss of property or even human life.

This section will present the derivation of partial differential equations that allow engineers to assess the amplitudes of free vibration of thin plates that are flexible enough to be simulated as thin membranes. Engineers may use this mathematical model for their modal analysis for safe design of these types of machine components and structures.

9.8.1 Derivation of the Partial Differential Equation

We will derive the mathematical model for the transverse vibration of thin plates with the following idealizations and assumptions:

1. The derivation of mathematical expressions is based on the lateral displacement of solids of plane geometry that are flexible and offer no resistance to bending. In reality, the structure corresponds to the description of "membranes."

2. The analysis considers thin plates with unsupported large spans that are sufficiently flexible in lateral deformation.

3. Because they are flexible, there is no shear stress in the thin plates in the subsequent analysis.

4. The thin plate is initially flat with its edges fixed. There is an initial sag represented by a function $f(x,y)$ sustained by in-plane tension P per unit length of the plate in all directions. The tension P is large enough to justify neglecting the weight of the plate.

5. Figure 9.23 shows the plate in the (x,y) plane with lateral displacement $z(x,y,t)$, the amplitude of vibration of the plate at the locations defined by the x–y coordinates and at time t.

6. Each element of the plate vibrates in the direction perpendicular to the plane surface of the plate in the z-coordinate as illustrated in Figure 9.24. The slopes of the deformed surface of the plate element are small.

7. The mass per unit area of the plate (m) is uniform throughout the plate.

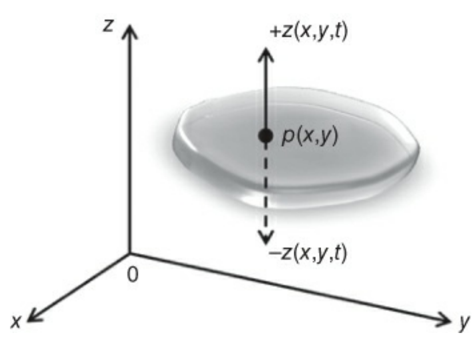


Figure 9.23 Lateral deformations of thin plates in vibration.

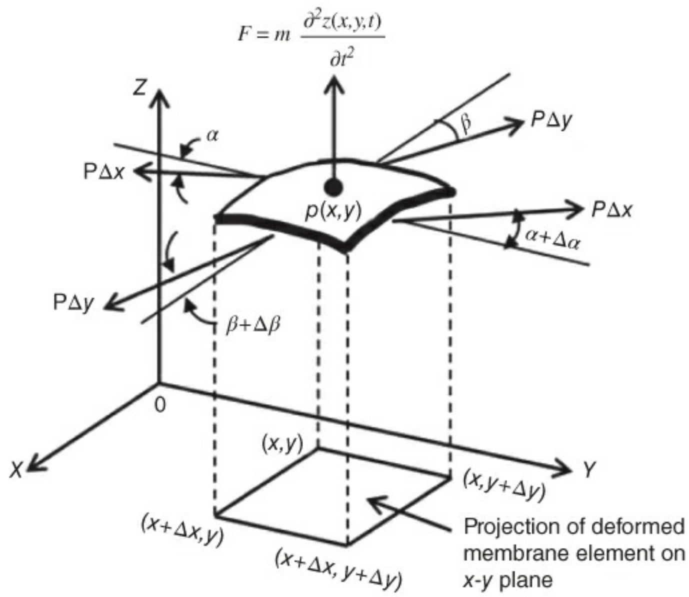


Figure 9.24 Forces on a vibrating element of a thin plate.

Figure 9.24 is a free-body diagram showing all the forces acting on a small element of the plate during a lateral vibration. The situation satisfies a dynamic equilibrium condition with the summation of all forces present at time *t* being equal to zero. Mathematically, we may express this condition in the form

$$\sum F_z = 0$$

The induced dynamic force *F* by Newton's second law plays a major role in the formulation of the above equilibrium of forces. Mathematically, this force may be expressed as

$$F = m\frac{\partial^2 z(x,y,t)}{\partial t^2}$$

as illustrated in Figure 9.24.

Referring to the small element shown in Figure 9.24, we have the following dynamic equilibrium conditions:

$$P\,\Delta x \sin(\alpha + \Delta\alpha) - P\,\Delta x \sin\alpha + P\,\Delta y \sin(\beta + \Delta\beta) - P\,\Delta y \sin\beta - m(\Delta x\,\Delta y)\frac{\partial^2 z[x,y,t]}{\partial t^2} = 0 \quad \textbf{9.75}$$

where m = mass per unit area of the plate material.

Since the slopes of the deformed plate element in both the x- and y-directions are very small as stipulated in “idealization number 6,” which means that both angles α and β are small, leading to the following approximate relationships:

$$\sin\alpha \approx \tan\alpha = \frac{\partial z\left(x+\frac{\Delta x}{2}, y, t\right)}{\partial y}$$

$$\sin(\alpha+\Delta\alpha) \approx \tan(\alpha+\Delta\alpha) = \frac{\partial z\left(x+\frac{\Delta x}{2}, y+\Delta y, t\right)}{\partial y}$$

$$\sin\beta \approx \tan\beta = \frac{\partial z\left(x, y+\frac{\Delta y}{2}, t\right)}{\partial x}$$

$$\sin(\beta+\Delta\beta) \approx \tan(\beta+\Delta\beta) = \frac{\partial z\left(x+\Delta x, y+\frac{\Delta y}{2}, t\right)}{\partial x}$$

Substituting the above approximate relationships into Equation 9.75 results in the following expression:

$$\begin{aligned} &P\,\Delta x\left[\frac{\partial z\left(x+\frac{\Delta x}{2}, y+\Delta y, t\right)}{\partial y} - \frac{\partial z\left(x+\frac{\Delta x}{2}, y, t\right)}{\partial y}\right] \\ &\quad + P\,\Delta y\left[\frac{\partial z\left(x+\Delta x, y+\frac{\Delta y}{2}, t\right)}{\partial x} - \frac{\partial z\left(x, y+\frac{\Delta y}{2}, t\right)}{\partial x}\right] \\ &\quad - m\,\Delta x\,\Delta y\frac{\partial^2 z(x,y,t)}{\partial t^2} \\ &\quad = 0 \end{aligned}$$

The following expression is obtained by dividing the above expression by $\Delta x \Delta y$:

$$\begin{aligned} &P\left[\frac{\frac{\partial z\left(x+\frac{\Delta x}{2}, y+\Delta y, t\right)}{\partial y} - \frac{\partial z\left(x+\frac{\Delta x}{2}, y, t\right)}{\partial y}}{\Delta y} + \frac{\frac{\partial\left(x+\Delta x, y+\frac{\Delta y}{2}, t\right)}{\partial x} - \frac{\partial z\left(x, y+\frac{\Delta y}{2}, t\right)}{\partial x}}{\Delta x}\right] \\ &\quad - m\frac{\partial^2 z(x,y,t)}{\partial t^2} \\ &\quad = 0 \end{aligned}$$

Given that the lateral deformation of the plate varies continuously with locations on the plane defined by the x- and y-coordinates, we have the following relationships:

$$\lim_{\Delta y \to 0} \frac{\frac{\partial z\left(x+\frac{\Delta x}{2}, y+\Delta y, t\right)}{\partial y} - \frac{\partial z\left(x+\frac{\Delta x}{2}, y, t\right)}{\partial y}}{\Delta y} = \frac{\partial^2 z(x,y,t)}{\partial y^2}$$

and

$$\lim_{\Delta x \to 0} \frac{\dfrac{\partial z\left(x + \Delta x, y + \dfrac{\Delta y}{2}, t\right)}{\partial x} - \dfrac{\partial z\left(x, y + \frac{\Delta y}{2}, t\right)}{\partial x}}{\Delta x} = \frac{\partial^2 z(x, y, t)}{\partial x^2}$$

The equilibrium equation thus has the following form with $\Delta x \to 0$ and $\Delta y \to 0$ for continuous variation of the amplitude of vibration of the plate in both x- and y-coordinates with

$$P\left[\frac{\partial^2 z(x, y, t)}{\partial y^2} + \frac{\partial^2 z(x, y, t)}{\partial x^2}\right] - m\frac{\partial^2 z(x, y, t)}{\partial t^2} = 0$$

or in the form

$$\frac{\partial^2 z(x, y, t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x, y, t)}{\partial x^2} + \frac{\partial^2 z(x, y, t)}{\partial y^2}\right] \quad \textbf{9.76}$$

in which the constant a in Equation 9.76 has the same *form* as in Equation 9.54 but a different meaning:

$$a = \sqrt{\frac{P}{m}} \quad \textbf{9.77}$$

in which P is the tension per unit length (with unit N/m), and m is the mass per unit area (kg/m^2). The constant a thus has the unit of m/s.

9.8.2 Solution of the Partial Differential Equation for Plate Vibration

Solution of the transverse vibration of a flexible plate in Equation 9.76 is far more complicated than that in Equation 9.54 for transverse vibration analysis for long flexible cables, for the reason of having additional variables in the equation for plate vibrations. We will use the following case, involving transverse vibration of a flexible plate as illustrated in Figure 9.25, to show how the separation of variables technique may be used to solve Equation 9.76.

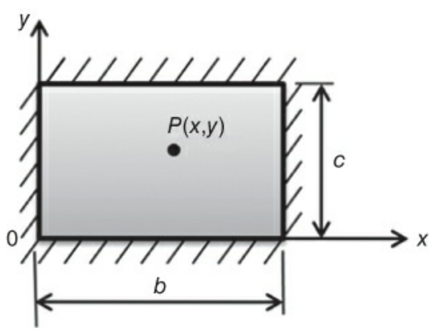


Figure 9.25 Plan view of a flexible plate undergoing transverse vibration.

The applicable partial differential equation is

$$\frac{\partial^2 z(x, y, t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x, y, t)}{\partial x^2} + \frac{\partial^2 z(x, y, t)}{\partial y^2}\right] \quad \textbf{9.76}$$

where the constant $a = \sqrt{P/m}$, P is the constant tension per unit length in the plate and m is the mass per unit area of the plate material. The following boundary and initial conditions apply for the case illustrated in Figure 9.25.

A. The boundary conditions:

$$z(x, y, t)|_{x=0} = z(0, y, t) = 0 \quad \textbf{a1}$$

$$z(x,y,t)|_{x=b} = z(b,y,t) = 0 \quad \textbf{a2}$$

$$z(x,y,t|_{y=0} = z(x,0,t) = 0 \quad \textbf{b1}$$

$$z(x,y,t)|_{y=c} = z(x,c,t) = 0 \quad \textbf{b2}$$

B. The initial conditions:

$$z(x,y,t)|_{t=0} = f(x,y) \quad \textbf{c1}$$

$$\left.\frac{\partial z(x,y,t)}{\partial t}\right|_{t=0} = g(x,y) \quad \textbf{c2}$$

where $f(x,y)$ is a specified function describing the initial shape of the plate; for example, the sag of the plate before the vibration takes place. The function $g(x,y)$ is another specified function that describes the velocity of the plate at the inception of the vibration. In many cases, $g(x,y) = 0$ for the plate to be initially in static equilibrium.

We begin our solution of Equation 9.76 by letting the amplitude of vibration of the plate to be a product of three individual functions:

$$Z(x,y,t) = X(x)\,Y(y)\,T(t) \quad \textbf{9.78}$$

in which $X(x)$, $Y(y)$, and $T(t)$ are functions involving independent variables x, y, and t, respectively.

We will obtain the following equality upon substituting Equation 9.78 into Equation 9.76, as we did in a similar way in Section 9.5.1. Such substitution yields

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2}$$

Inspection of the above expression reveals that the left-hand side of the equality relates to variable x only, whereas that variable x does not appear in the right-hand side of the equality. The only way such an equality can be valid is for both sides of the equality to equal the same constant. As is a general practice when using the separation of variables technique, a negative sign is assigned to the constant to avoid unbounded solutions of $z(x,y,t)$. We thus have the following relationships with the first separation constant λ:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} = -\lambda^2 \quad \textbf{d}$$

These equalities lead to the following differential equation:

$$\frac{d^2X(x)}{dx^2} + \lambda^2X(x) = 0 \quad \textbf{e}$$

and another equality:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} + \lambda^2 \quad \textbf{f}$$

Using the same argument as we did with the equality in Equation (d), we derive the following equalities for functions involving the independent variables y and t:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} + \lambda^2 = -\mu^2 \quad \textbf{g}$$

where μ is second separation constant used in separating the variables y and t in Equation (f). We may derive two additional differential equations from the above equalities:

$$\frac{d^2Y(y)}{dy^2} + \mu^2 Y(y) = 0 \tag{h}$$

and

$$\frac{d^2T(t)}{dt^2} + a^2(\lambda^2 + \mu^2)T(t) = 0 \tag{j}$$

We recognize that all three differential equations for $X(x)$, $Y(y)$, and $T(t)$ in the respective Equations (e), (h), and (j) are linear second-order differential equations with constant coefficients. Solutions for these functions are available in Section 8.2. The general solutions of these functions can be expressed as follows:

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \tag{k1}$$

$$Y(y) = c_3 \cos \mu y + c_4 \sin \mu y \tag{k2}$$

$$T(t) = c_5 \cos a\sqrt{\lambda^2 + \mu^2}\, t + c_6 \sin a\sqrt{\lambda^2 + \mu^2}\, t \tag{k3}$$

The constants c_1, c_2, c_3, ..., c_6 in Equations (k1), (k2), and (k3) are arbitrary constants to be determined by appropriate specified conditions in Equations (a1), (a2), (b1), (b2), (c1), and (c2).

The following conditions for the determination of constants in Equations (k1) and (k2) are derived by substituting the relationship in Equation 9.78 into Equations (a1), (a2), (b1), and (b2), yielding the following conditions for Equations (k1) and (k2). $X(0) = 0$ and $Y(0) = 0$ lead to $c_1 = c_3 = 0$. The other two conditions: $X(b) = 0$ and $Y(c) = 0$ lead to the expressions for the separation constants: $\lambda = m\pi/b$ with $m = 1, 2, 3, \ldots$, and $\mu = n\pi/c$ with $n = 1, 2, 3, \ldots$. We thus have the solutions of $X(x)$ and $Y(y)$ as follows:

$$X(x) = c_2 \sin \lambda x = c_{2,m} \sin \frac{m\pi x}{b} \qquad \text{with} \quad m = 1, 2, 3, \ldots \tag{m1}$$

$$Y(y) = c_4 \sin \mu y = c_{4,n} \sin \frac{n\pi y}{c} \qquad \text{with} \quad n = 1, 2, 3, \ldots \tag{m2}$$

The constant c_5 and c_6 in solution in Equation (k3) may be determined with the initial conditions specified in Equations (c1) and (c2) for $z(x,y,t)$:

$$\begin{aligned} z(x,y,t) &= X(x)\,Y(y)\,T(t) \\ &= \left(c_{2,m} \sin \frac{m\pi x}{b}\right)\left(c_{4,n} \sin \frac{n\pi y}{c}\right) \\ &\quad \left(c_5 \cos a\sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2}\, t + c_6 \sin a\sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2}\, t\right) \end{aligned} \tag{n}$$

Let

$$\omega_{mn} = \sqrt{\lambda_m^2 + \mu_n^2} = \sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2} \tag{p}$$

The solution $z(x,y,t)$ may be expressed in the form

$$z(x,y,t) = A_{mn}\left(\sin \frac{m\pi x}{b}\right)\left(\sin \frac{n\pi y}{c}\right)(A_{mn} \cos a\omega_{mn} t + B_{mn} \sin a\omega_{mn} t) \tag{q}$$

with $m = 1, 2, 3, \ldots$, and $n = 1, 2, 3, \ldots$.

One would have observed that we made the products of constants $(c_{2,m})(c_{4,n})(c_5) = A_{mn}$ and $(c_{2,m})(c_{4,n})(c_6) = B_{mn}$ in Equation (p).

Since the solution in Equation (q) is valid for every value of m and n, the complete solution will be the sum of all valid solutions as expressed in the following:

$$z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\sin \frac{m\pi}{b} x\right) \left(\sin \frac{n\pi}{c} y\right) (A_{mn} \cos a\omega_{mn} t + B_{mn} \sin a\omega_{mn} t) \tag{9.79}$$

The multi-valued constant coefficients A_{mn} and B_{mn} in Equation 9.80a may be determined using the two initial conditions in Equations (c1) and (c2) following a similar procedure to that outlined in determining coefficient b_n in Section 9.7.2. However, in the current situation, we need to determine A_{mn} by multiplying the double summation in Equation 9.80a by the product $\sin(m\pi x/b)\sin(n\pi y/c)$ on both sides of the equation and then integrating over the ranges $0 \le x \le b$ and $0 \le y \le c$. This will give the following expressions for these two multi-valued constant coefficients:

$$A_{mn} = \frac{4}{bc} \int_0^c \int_0^b f(x, y) \left(\sin \frac{m\pi}{b} x\right) \left(\sin \frac{n\pi}{c} y\right) dx\, dy \tag{9.80a}$$

and

$$B_{mn} = \frac{4}{abc\omega_{mn}} \int_0^c \int_0^b g(x, y) \left(\sin \frac{m\pi}{b} x\right) \left(\sin \frac{n\pi}{c} y\right) dx\, dy \tag{9.80b}$$

The transverse vibration modes of flexible plates may be obtained by assigning m = 1 and n = 1 for mode 1, and m = 2 and n = 2 for mode 2 vibrations, and so on. The periods of the various modes of vibration may be computed from the expression: $T_n = 2\pi/\omega_{mn}$, with ω_{mn} as given by Equation (p).

9.8.3 Numerical Solution of the Partial Differential Equation for Plate Vibration

Numerical solution for the amplitude of transverse vibration of flexible plates given in Equation 9.79 with coefficients in Equations 9.80a and 9.80b is much more tedious and complicated than that for flexible cables described in Section 9.7.

We will present the finding of such a solution using a commercial software package called MatLAB for a case of free-vibration analysis of a flexible mouse pad such as shown in Figure 9.25 with dimensions b = 10 inches and c = 5 inches. The pad has thickness 0.185 inch and mass density 1.55 lb_m/in^3. The pad has an initial shape that can be described by the function $f(x, y) = xy(b - x)(c - y)$, or $f(x, y) = xy(10 - x)(5 - y)$, and is subjected to a constant tension of $P = 0.5\ \text{lb}_f/\text{in}$. Vibration of the pad begins from a state of static equilibrium, *with* $g(x, y) = 0$ in Equation (c2) in Section 9.8.2.

We first determine the constant coefficient a in Equation 9.77:

$$a = \sqrt{\frac{Pg}{\rho}} = \sqrt{\frac{(0.5\ \text{lb}_f/\text{in})(32.2\ \text{ft/s}^2)}{0.00155\ \text{lb}_m/\text{in}^2} \left(\frac{12\ \text{in}}{\text{ft}}\right)} = 353.05\ \text{in/s}$$

The frequency ω_{mn} required to compute the period T is given in Equation (p) in Section 9.8.2 with eigenvalues $\lambda_m = m\pi/10$ and $\mu_n = n\pi/5$ with m, n = 1, 2, 3, ..., as shown in the same section.

The mode shapes of this plate are computed from free lateral vibration analysis with the following expression:

$$z(x, y, t) = A_{mn} \left(\sin \frac{m\pi x}{10}\right) \left(\sin \frac{n\pi y}{5}\right) (A_{mn} \cos a\omega_{mn} t\,) \tag{9.81}$$

in which $m, n = 1, 2, 3, ...$

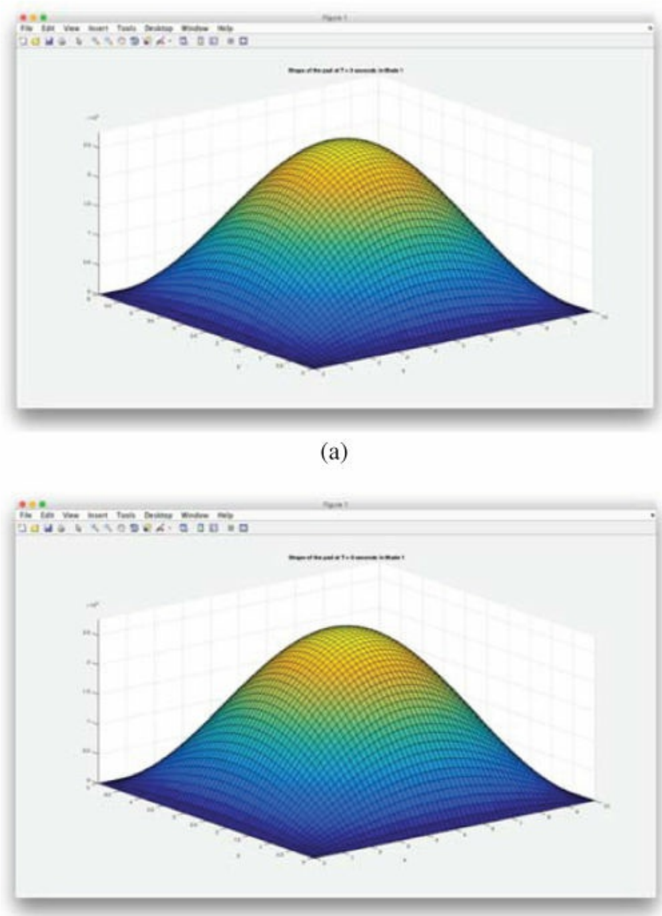
The coefficient A_{mn} in the above expression has the form

$$A_{mn} = \frac{16(bc)^2[1 + (-1)^{n+1}][1 + (-1)^{m+1}]}{m^3 n^3 \pi^6} \quad \textbf{9.82}$$

with $b = 10$ inches and $c = 5$ inches.

Graphical representations of this numerical solution of the modal analysis of a flexible mouse pad are obtained with the MatLAB software package (version R2015a) made available by the College of Engineering at San José State University. An overview of this software will be given in Section 10.6.2 of Chapter 10, with input/output files for this analytical problem presented in Case 2 in Appendix 4.

Figure 9.26 shows the shape of the pad in mode 1 vibration with $m = n = 1$ in Figure 9.26a at time $t = 0$ and Figure 9.26b at $t = 1/8$ second. There is a significant drop in the maximum amplitude during this period of time.



(a)

(b)

Figure 9.26 Shape of the mouse pad in mode 1 vibration. (a) The initial shape of the pad. (b) Mode shape at time 1/8 second.

Figure 9.27 shows graphically the shape of the pad in mode 3 vibration with $m = n = 3$ in three instants of $t = 0$, 1/8, and ¼ seconds. We observe that the pad will have three multiple peaks of vibrations along the x- and y-directions, respectively. Mode shapes and peak amplitudes of this mode at different instants are depicted in the same figure.

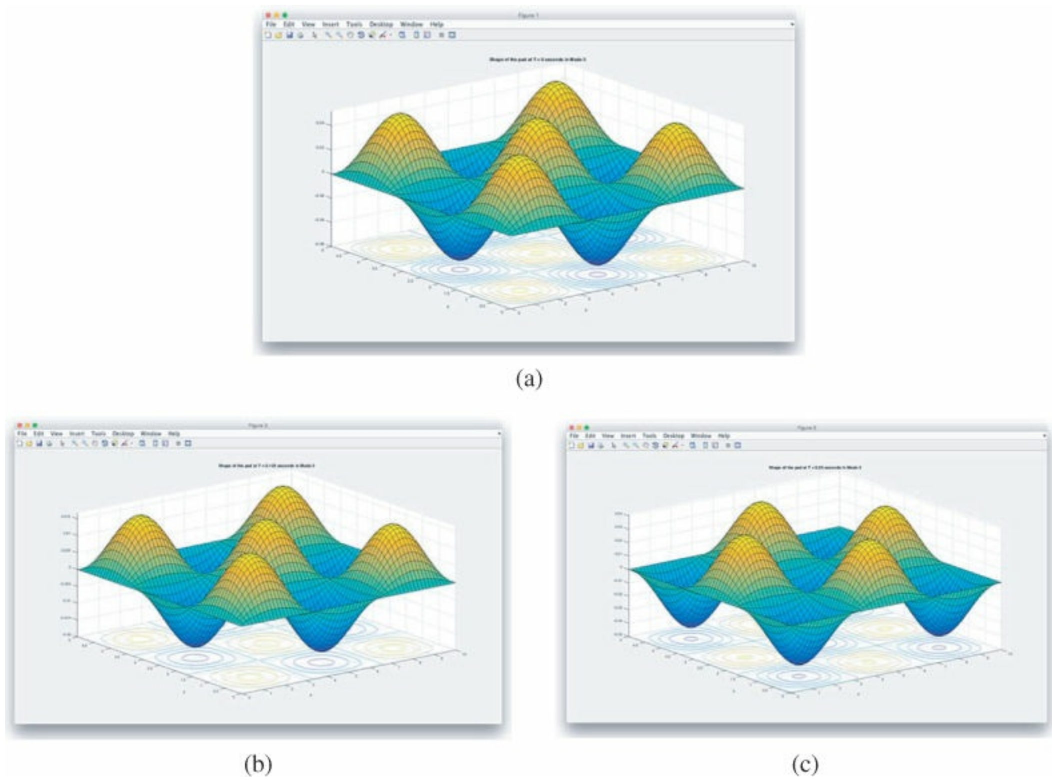


Figure 9.27 Mode 3 shapes in vibration of a flexible plate. (a) At time t = 0. At time t = 1/8 second. (c) At time t = 1/4 second.

As described in Section 8.9, when engineers design a thin plate structure such as the printed circuit board, mode shapes obtained from modal analyses would allow them to avoid placing delicate electronic components at locations where maximum amplitude of vibration may occur such as that illustrated in Figures 9.26 and 9.27.

Problems

9.1 Determine the first- and second-order partial derivatives of each of the following functions:

a. $f(x, y) = 5x^3 + 10x^2y + 8xy^2 + 7y^3$

b. $x(t) = \frac{1}{10}\cos 10t + \frac{1}{200}\sin 10t - \frac{t}{20}\cos 10t$

c. $u(x, t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$, where b_n (n = 1, 2, 3,..., n) are constants.

9.2 Use the Fourier transform in Section 9.3.3 to solve Equation 9.54 for the amplitudes of vibration of a long flexible cable:

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

with $-\infty < x < \infty$ and $t > 0$ and where a is a constant. The specified initial conditions are

$$u(x,t)\big|_{t=0} = u(x,0) = f(x)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = u'(x,0) = g(x)$$

9.3 Determine whether the separation of variables technique may be used to solve the following partial differential equations:

a. $\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t}$

b. $\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} - u(x,t) = 0$

c. $\frac{\partial^2 \varphi(x,y)}{\partial x^2} + \frac{\partial^2 \varphi(x,y)}{\partial x \partial y} + \frac{\partial^2 \varphi(x,y)}{\partial y^2} = 0$

d. $\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} + a^2 \qquad (a = \text{constant})$

9.4 Determine whether the temperature distribution in a long metal rod such as illustrated in Figure 7.16 can be obtained by solving the following differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

in which $\alpha = k/(\rho c)$ is the thermal diffusivity of the rod material, where k = thermal conductivity in W/cm-°C, ρ = mass density in g/cm^3, and c = specific heat of the material in J/g-°C. Thermal diffusivity α for most engineering materials in normal temperature is a constant.

The prescribed conditions for this problem are

Initial condition (IC): $T(x,0) = f(x) = 100 - 40x$, a given function.

End conditions (BC): $T(0,\ t) = 0°\text{C}$ and $T(L,t) = 0°\text{C}$,where L is the length of the rod.

A numerical case involving L = 1 m is to be generated with the following material properties of copper:

Thermal conductivity $k = 3.93$ W/cm-°C

Mass density $\rho = 8.9$ g/cm^3

Specific heat $c = 0.386$ J/g-°C

Use the separation of variables technique to solve for the temperature $T(x,t)$ by letting $T(x,t) = X(x)\tau(t)$ where $X(x)$ and $\tau(t)$ are two distinct functions. Derive the equations for these two functions with appropriate conditions.

9.5 A rod of length L is made of a material with thermal conductivity k, initially with a temperature distribution along its length (the x-direction) that can be described by a function $f(x)$. Find the temperature distribution in the rod $T(x,t)$ at time $t > 0$ if all the surfaces and the two ends of the rod are thermally insulated.

9.6 Write the differential equation with appropriate conditions for the case in Problem 9.5 but with its circumferential surface at $r = b$ exposed to surrounding air with bulk temperature T_a and heat transfer coefficient h.

9.7 A rectangular plate used as a heat spreader with the temperatures at its four sides maintained as indicated in Figure 9.10. The temperature distribution in the plate $T(x,y)$ may be obtained by solving the partial differential equation

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0 \qquad \textbf{9.50}$$

a. Show the mathematical forms of the boundary conditions required to solve Equation 9.50 according to the specified conditions indicated in Figure 9.28.

b. Solve Equation 9.50 with the specified boundary conditions in (a).

c. Determine the temperature at the center of the plate.

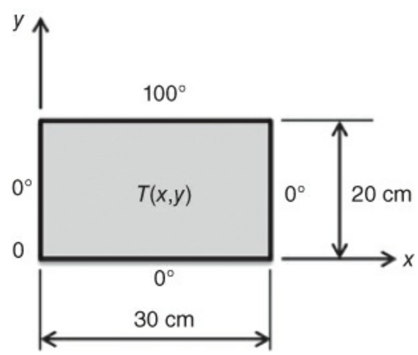


Figure 9.28 Temperature distribution in a flat plate.

There is negligible heat flow from the flat faces of the plate in contact with stagnant air. All temperatures in Figure 9.28 have the unit of °C.

9.8 Solve Equation 9.54 for a long cable of length L in free vibration as illustrated in Figure 9.18 with the initial shape of the cable being $f(x) = x(L - x)$.

9.9 Do the same as in Problem 9.8 but with $f(x) = \sin(\pi x)/L$.

9.10 The amplitude $y(x,t)$ of free vibration of a beam of length L can be obtained by solving the following partial differential equation (called the Euler–Lagrange equation):

$$a^2 \frac{\partial^4 y(x,t)}{\partial x^4} = \frac{\partial^2 y(x,t)}{\partial t^2} \qquad \text{for} \quad 0 \le x \le L \quad \text{and} \quad t > 0$$

where $a^2 = EI/m$, in which E is the Young's modulus and m is the mass per unit length of the beam material, and I is the section moment of inertia of the beam cross-section.

Solve the partial differential equation with the following conditions. The beam is simply supported with no bending moments (bending moments are proportional to the second order derivative of the deflection of the beam) and the initial conditions of the beam are specified by the following functions of $f(x)$ and $g(x)$:

$$y(x,t)|_{t=0} = y(x,0) = f(x)$$

and

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = g(x)$$

9.11 Solve Problem 9.10 with $f(x) = \sin(\pi x)/L$.

9.12 Solve Problem 9.11 but with both ends of the beam being rigidly held; that is, the slope of the beam at both ends is zero.