Chapter 2
Mathematical Modeling

Chapter Learning Objectives

- Learn what mathematical modeling is and its application in engineering analysis.
- Learn the physical representation of mathematical entities such as functions, variables, derivatives, and integrals in engineering analysis.
- Understand continuous functions and functions with discrete values.
- Understand curve fitting by polynomial functions.
- Appreciate the application of derivatives in engineering analysis.
- Appreciate the application of integrals in engineering analysis.
- Learn special functions in engineering analysis.
- Understand differential equations in engineering analysis and how they are derived.

2.1 Introduction

As was indicated in Section 1.1 in Chapter 1, “mathematical modeling” is used as a principal tool in engineering analysis. It is reasonable to ask “What is mathematical modeling?” Different answers to the question will be given by different people. Our definition of mathematical modeling is that it is “an act of translating back and forth between mathematics and physical situations.”

An analogy to the above definition might be a beautiful melody being developed in the mind of a composer with his or her subsequent expressing this melody in symbolic notes on the stave, as illustrated in Figure 2.1

Figure 2.1 Translation of a melody into music notes—an analogy to mathematical modeling in engineering.

In engineering analysis, however, the role that mathematical modeling plays is significantly more complicated than that implied in Figure 2.1. Figure 2.2 illustrates that mathematical modeling involves most of the tasks stipulated in Stages 2 and 3 of engineering analyses as described in Section 1.4 in Chapter 1.

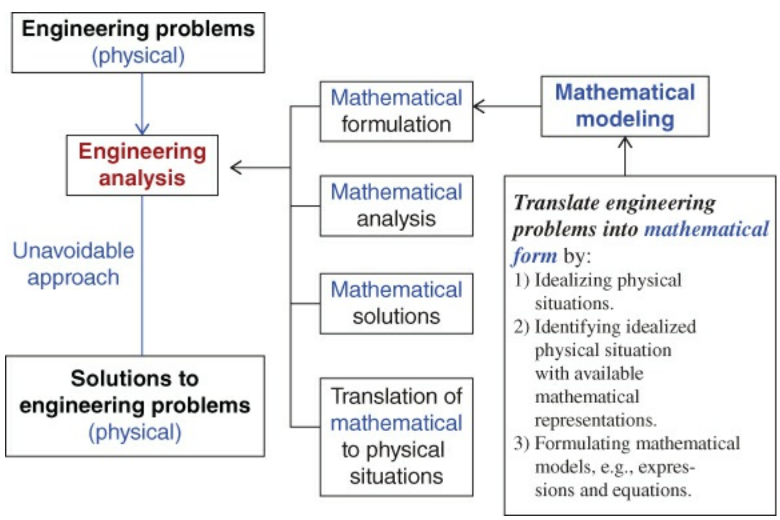

Figure 2.2 The role of mathematical modeling in engineering analysis.

There are essentially four different forms of math tools available to engineers in their engineering analyses.

Form 2.1 Empirical formulas

Empirical formulas are developed to describe physical situations that are too complicated to be expressed in closed-form mathematical expressions. These formulas contain factors that can only be determined by experiment. Many of these formulas are derived from engineers' own experience, or their employers.

There are types of empirical formulas that are frequently used in engineering analysis. The following are two examples that engineers may use in their analyses.

1. Estimation of the pressure drop in fluid flow in pipes

Pressure drop (ΔP) is a primary parameter for engineers in their design of fluid flow in pipes and tubes and occurs in many thermal/fluid engineering systems analyses. Fluid flow is possible only if the supply pumping energy is sufficient enough to overcome the pressure drop along its flow path. Pressure drop in pipe flow is primarily induced by friction between the pipe (or tube) walls and the contacting fluids.

Several empirical formulas are available for estimating the pressure drop in straight pipe flow, such as that shown in Equation 2.1 (Moody, 1944):

$$\Delta P = f\frac{\rho v^2}{2}\frac{L}{D} \tag{2.1}$$

where ρ = mass density of the fluid, v = average velocity of fluid flow, L = length of the pipe, D = diameter of the pipe, and f = friction factor from Moody chart (http://en.wikipedia.org/wiki/Moody_chart).

The friction factor f in Equation 2.1 with the values available from the Moody chart is derived from experiment.

Additional pressure drop occurs when the fluid flows through bends in a pipeline (see “Bends, flow and pressure drop in” at http://www.thermopedia.com/content/577/). This additional pressure drop in the bends is induced by a radial pressure gradient created by the centrifugal force acting on the fluid in the flow. The following empirical formula is used for estimating the pressure drop in pipe bends:

$$\Delta P = f\frac{\rho v^2}{2}\frac{\pi R_b}{D}\frac{\theta}{180°} + \frac{1}{2}k_b \rho v^2 \tag{2.2}$$

where f is the same Moody friction factor as shown in Equation 2.1, R_b = radius of the pipe bend, θ = the angle of the bend, and k_b = the bend loss coefficient.

Numerical values of the bend loss coefficient k_b in Equation 2.2 are available in handbooks (e.g., Friend and Idelchik, 1989), or from plots such as that in Figure 2.3.

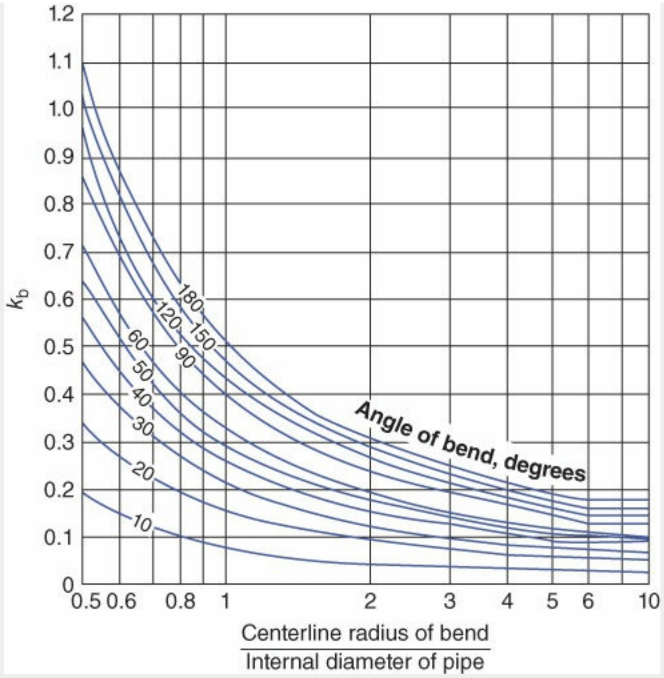

Figure 2.3 Head loss of fluids passing pipe bends.

2. Empirical formulas in convective heat transfer analysis

Heat transfer by convection, in particular, forced convective heat transfer is used in the design analysis of many items of heat transfer equipment such as tubular and compact heat exchangers and steam generators. Physical situations in such analyses involve convective heat transfer between the hot and cold fluids separated by tubular containing surfaces such as illustrated in Figure 2.4. Effective heat transfer between the two fluids is achievable with motion of the fluids. The pressure drop (ΔP) appearing in Equations 2.1 and 2.2 thus plays a significant role in this type of analysis.

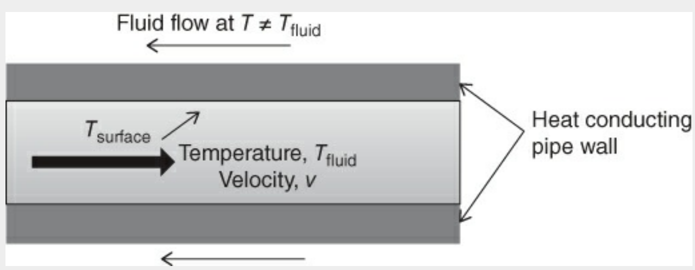

Figure 2.4 Convective heat transfer between two fluids at different temperatures.

The amount of heat transfer from the hot fluid to the cold in Figure 2.4 is characterized by a heat transfer coefficient *h*, which appears in the newtonian cooling law that governs convective heat transfer (Kreith and Bohn, 1997):

$$q_c = h A (T_{surface} - T_{fluid}) \quad \textbf{2.3}$$

where q_c = heat transferred from the temperature at the wall surface ($T_{surface}$) of the tube to the temperature of the fluid (T_{fluid}) flowing inside the tube. A = the contacting surface area of the fluid and the tube wall.

Numerical values of the heat transfer coefficient h in Equation 2.3 are too complicated to be determined by closed-form formulas. The following expression in Equation 2.4 is used for the evaluation of the heat transfer coefficient h in Equation 2.3:

$$\mathrm{Nu} = \alpha(\mathrm{Re})^{\beta} (\mathrm{Pr})^{\gamma} f\left(\frac{x}{d_e}\right) \quad \textbf{2.4}$$

where Nu = average Nusselt number = (hD/k), Re = Reynolds number = ($\rho Dv/\mu$), and Pr = Prantl number = ($\mu c_p/k$), and the coefficient α and indices β and γ are constants determined by experiment. Other symbols in Equation 2.4, ρ, k, c_p, and μ are the respective mass density, thermal conductivity, specific heat under constant pressure and dynamic viscosity of the fluid inside the pipe, and D is the average pipe diameter.

The function $f(x/d_e)$ in Equation 2.4 relates to the aspect ratio of the pipe cross-section, with x indicating the location of the cross-section, and the hydraulic diameter located at x is expressed by

$$d_e = \frac{4A}{p}$$

in which A is the cross-sectional area of the fluid flow and p is the wet perimeter.

This hydraulic diameter d_e is used for either partially filled or fully filled flow in the cross-section of a pipe of a cross-sectional given geometry.

Thus, the value of heat transfer coefficient, h can be obtained from the Nusselt number in terms of the Reynolds and Prantl numbers computed with experimentally determined coefficients and parameters in Equation 2.4.

Form 2.2 Algebraic equations

Many textbooks, handbooks (Avallone et al., 2006; Kreith, 1998; Bishop, 2002; Gad-el-Hak, 2002; Whitaker, 1996; Gibilisco, 1997), and technical papers offer simple algebraic formulas and expressions for engineering analysis. Typical examples are reported as “stresses and deflection of beam structures subjected to bending loads, dynamic forces analysis” from engineering mechanics (Meriam and Kraige, 2007), and “temperature and heat flow in solids or fluids” from heat transfer textbooks (Kreith and Bohn, 1997; White, 1994; Janna, 1993). Equations (1.1) to (1.4) given in Chapter 1 are typical algebraic equations used in mathematical modeling for simple structure design analysis.

Form 2.3 Differential and integral equations

These equations are derived from the laws of physics for applications in engineering analysis. The following are typical examples of differential and integral equations for such applications

1. Differential equation in diffusion analysis

The following differential equation (2.5) is used to determine the concentration of a foreign substance such boron ions, represented by $C(x,t)$, diffusing into another base substance such as silicon substrate—a case of analysis of doping of semiconductor (Hsu, 2008):

$$\frac{\partial C(x,t)}{\partial t} = D\,\frac{\partial^2(x,t)}{\partial x^2} \quad \textbf{2.5}$$

in which $C(x,t)$ is the concentration of the foreign substance at depth x in the base substance at time t; D is the diffusivity of the specific foreign substance in the base substance, a given material characteristic available from handbook.

2. Integral equation in mathematical modeling of engineering analysis

Often, engineers need to solve for the function $f(x)$ that represents the solution of the problem embedded in an equation that takes the form of an integral as presented in Equation (2.6). (Spiegel, 1963):

$$\int_0^\infty f(x)\cos ax\,dx = \left\{\frac{1-\alpha \quad 0 \le \alpha \le 1}{0 \quad \alpha > 1}\right. \quad \textbf{2.6}$$

The solution of this integral equation is

$$f(x) = \frac{2(1-\cos x)}{\pi x^2}$$

Form 2.4 Numerical solution methods

There are ample occasions when mathematical modeling for engineering analysis become too complicated, with the requirement for solving nonlinear equations such as transcendental equations in solutions of partial differential equations, as will be presented in Chapter 9, as well as nonlinear differential and integral equations. Closed-form solutions will not be available for these cases. Numerical analysis techniques that require the use of digital computer with algorithms developed specifically for these types of problems are then the only viable technique for solution. There are a number of sources that engineers may use for their numerical analysis, for example[Burden and Faires (2011) and Sauer (2011). Some numerical methods for engineering analysis will be presented in Chapters 10 and 11.

As demand arises for sophisticated engineering analysis on increasing numbers of cases involving complex physical conditions in the geometry, loading, and boundary conditions—such as thermofracture, thermohydraulic, and thermomechanical analyses that require advanced engineering analysis—the forms 1 to 3 of mathematical modeling just outlined are no longer adequate. The finite-element method (FEM) and finite-difference method (FDM) are the appropriate tools for such analyses. Both of these methods are based on discretizing bodies of complex geometry into a finite number of elements of specific simple geometry interconnected at nodes of the elements. Analyses are performed on these elements rather than on the whole body. Publications describing the principles of these advanced engineering analysis techniques are available in the literature (Turner et al., 1959; Zienkiewicz, 1971; Bathe and Wilson, 1976; Hsu, 1986; LeVeque, 2007). There are several commercial software packages available to engineers for advanced engineering analysis of virtually every discipline. Chapter 11 will cover the basic principles and formulations of stress analysis using the finite-element method.

2.2 Mathematical Modeling Terminology

We will begin mathematical modeling by reviewing the physical meanings of some of the terminology that is frequently used in engineering analyses. It is important for engineers to relate many of the terms that they encountered in previous mathematical courses with the physical meanings these terms represent in solving engineering problems. In this section, we will learn what roles such common terms as “function” and “variable” play in engineering analysis, as well as the physical meanings of “differentiation” and “derivative” in the analysis. We will also learn what “integration” is in a physical sense, and how this mathematical operation can help in solving many engineering problems.

2.2.1 The Numbers

2.2.1.1 Real Numbers

Real numbers can either be integers or rational numbers—a, b, a/b, and so on, with the numbers a and b being constants. Rational numbers can be integers or fractional numbers.

2.2.1.2 Imaginary Numbers

These are the numbers that are multiplies of square root (−1) and that, along with real numbers, compose the complex numbers. Imaginary numbers do not have physical meaning but they appear in some mathematical expressions and solutions.

2.2.1.3 Absolute Values

The absolute value of a number recognizes the value (“magnitude”) but not the sign attached to the value. Mathematically, the absolute value can be expressed as

$$\begin{aligned} |x| &= x \qquad \text{meaning } x = 0 \text{ or a positive value} \\ &= -x \qquad \text{meaning } x \text{ is negative, i.e. } x < 0 \end{aligned}$$

2.2.1.4 Constants

For a constant the value of the number does not change and is always fixed.

2.2.1.5 Parameters

When the value of a number is treated as a “constant” under specific conditions or circumstances we refer to it as a parameter.

2.2.2 Variables

The value of a variable varies with physical conditions, mainly in terms of spatial position or with time. There can be more than one variable involved in engineering analysis.

Two types of variables are commonly involved in engineering analysis:

1. *Spatial variables*. These designate positions in a given space, e.g., (x,y,z) in a space defined by a rectangular coordinate system, or (r,θ,z) in a cylindrical polar coordinate system.
2. *Temporal variable*. The variable representing variation of a physical quantity or system with time t.

Both these types of variables are called “independent variables” because variation of any of these individual variables x, y, and z, or r, θ, z, and t do not affect the values of the other variable presented in the same cases.

2.2.3 Functions

Functions are normally used to represent the physical properties in engineering analyses. Specific functions may be represented with the same symbols used to denote various physical quantities in the analysis. The functions or physical quantities may be represented in mathematical modeling with Latin or Greek letters, according to convention; typical notation includes the following:

Mass (m), weight (W), length (L), area (A), and volume (V) of a solid.

Forces applied to a solid (F).

Temperature (T) of substances, e.g., temperature of a solid or fluid.

Velocity of a rigid solid body or a fluid in motion (V).

Distance that a rigid body travels in a straight or curved path (S).

Stress (σ) and strain ( ) in deformed solids.

The typical mathematical expression of functions such as those designating the physical quantities listed above, usually involve associated variables. For example, the function $F(x)$ may represent the physical quantity of force F acting on a beam at a distance x from a reference point. The value of F depends on the value of the variable x in this case.

Change of the value (or magnitude) of function F associated with change in the values of the variable (or variables) of the function can take three different forms.

2.2.3.1 Form 1. Functions with Discrete Values

In general, the value of a function is determined by the values of all variables associated with the function. The values of all the variables may vary independently of all other variables associated with the function. The values of functions with discrete values are determined by the values of discrete variables. For example, in a hypothetical case that involves a family of four (father, mother, and two children) standing on a beam as illustrated in Figure 2.5a, the load to the beam involves four equivalent concentrated forces corresponding to the weights of the members of this family defined by where the individual members stand. The function $W(x)$ represents the loading forces to the beam with the variable x being the location of the beam, at which the load W is accounted for. Thus, the function $W(x)$ has four discrete values W_1 at $x = x_1$, W_2 at $x = x_2$, W_3 at $x = x_3$, and W_4 at $x = x_4$ as shown in Figure 2.5b.

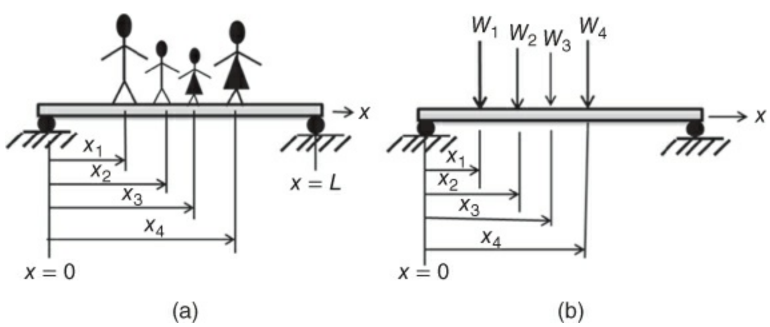

Figure 2.5 Discrete forces applied to a beam structure. (a) People standing on a beam. (b) Equivalent forces applied to the beam.

2.2.3.2 Form 2. Continuous Functions

Because the value of a function is determined by the value of the associated variable or variables, the value of the function can vary *continuously* with the variation of the associated variable(s), as illustrated in Figure 2.6 for the ambient temperature variation at a place over a period of a day; in the function T will represent the ambient temperature at the time of day, which is denoted by t. The value of the function $T(t)$ varies continuously with continuous variation of the value of the variable t.

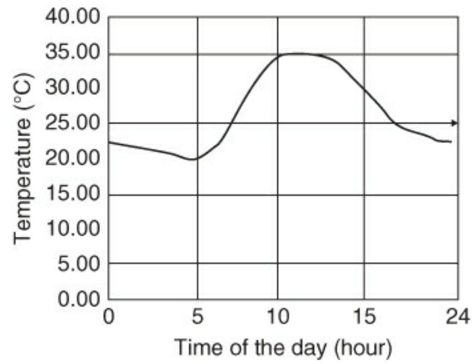

Figure 2.6 Typical diurnal variation of the ambient temperature of a location.

Often a function of discrete values may be approximated by a continuous function if the values of the discrete function are small increments of the associated variable, such as in the case illustrated in Figure 2.7, where the weight *W* applied to the beam by people standing close together on the beam is as shown in Figure 2.7a. The function representing the equivalent forces *W* applied to the beam may be approximated by an approximated continuous function *W*(*x*) as shown in Figure 2.7b.

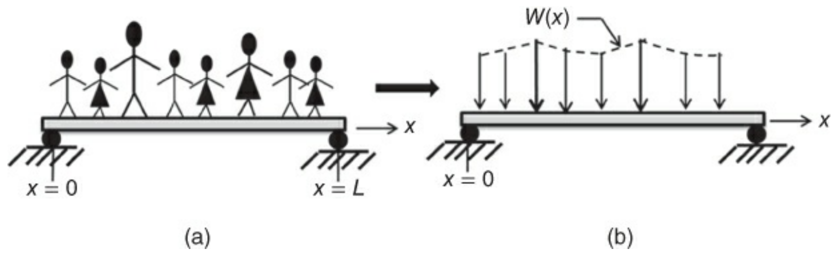

Figure 2.7 Discrete and approximated continuous functions. (a) People standing close to each other. (b) Approximate continuous variation of forces.

2.2.3.3 Form 3. Piecewise Continuous Functions

The magnitudes of the physical quantities associated with many physical phenomena in engineering practice vary in ways that can be represented by either continuous functions, as illustrated in Figure 2.6, or by functions of discrete values as illustrated in Figure 2.5. However, there are other cases; one may observe, for example, that the instantaneous water level (y) in the office water cooler tank shown in Figure 2.8 may be represented by both discrete values and continuous variation with the variable time (t). Graphical representation of this physical phenomenon is shown in Figure 2.9, which shows how the water level drops intermittently over time after each serving of water. It illustrates a continuous drop of water level while the release tap is open, followed by a period of constant water level before the next serving.

Figure 2.8 A typical water cooler tank.

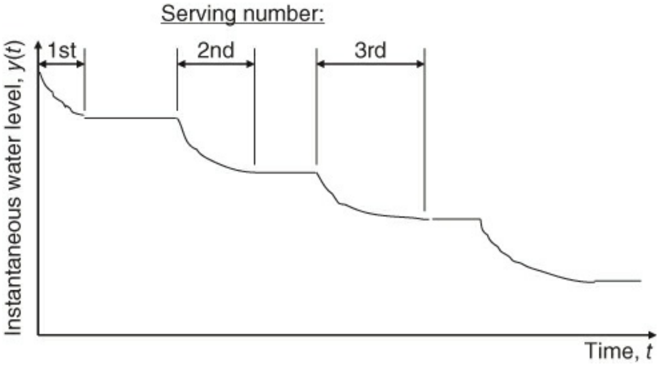

Figure 2.9 A qualitative representation of water level in the drinking water tank at different times.

In many cases, functions may involve more than one independent variable, e.g., a function $F(x,y)$, in which the value of function F varies with independent variations of the two independent variables, x and y.

In summary, we observe the following properties of a function:

1. Functions are dependent variables themselves because the value of a function changes depending on the values of its associated variables.
2. The independent variables are spatial variables or temporal variables, or both.

2.2.4 Curve Fitting Technique in Engineering Analysis

Curve fitting in engineering analysis is undertaken to derive a continuous function that will best fit either a set of data points from experiments or functions with discrete values such as that illustrated in Figure 2.7b. This technique is frequently used to describe the geometric profile of solids in engineering analyses. The continuous functions that are derived using a curve fitting technique can also be used for "interpolation" and "extrapolation" of the value of the function within and outside the range of the variables used in defining this function.

A number of techniques are available for performing curve fitting. Popular techniques used in engineering analysis include (1) least-squares technique, (2) spline curve fitting technique, and (3) polynomial curve fitting technique.

The least-squares curve fitting technique (https://en.wikipedia.org/wiki/Least_square) results in more accurate fitting of large numbers of widely scattered data, whereas the function that is derived using the spline fitting technique (https://en.wikipedia.org/wiki/Spline_(mathematics)) passes through all the data points with smooth transitions. Use of the polynomial function for curve fitting is the simplest of all curve fitting techniques, and will be described in the next subsection.

2.2.4.1 Curve Fitting Using Polynomial Functions

This technique involved the derivation of a polynomial function of order n that will pass through all the given data or sample points. Figure 2.10 shows a data set of five temperatures measured in a production process.

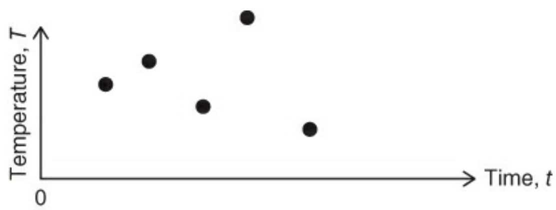

Figure 2.10 Measured temperatures from a fabrication process.

We will use this example to derive a polynomial function that will pass through all measured data points by following a procedure that begins with the assumption of a polynomial function of order n in the general form

$$Y(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n \qquad \textbf{2.7}$$

in which the order of the polynomial function $n = N - 1$, with N being the total number of available data (sample) points. The unknown coefficients $A_0, A_1, A_2, A_3, \ldots, A_n$ are determined by a set of simultaneous equations established from the given coordinates of the sample points.

The coordinate system, x–y used in Equation 2.7 is shown in Figure 2.11.

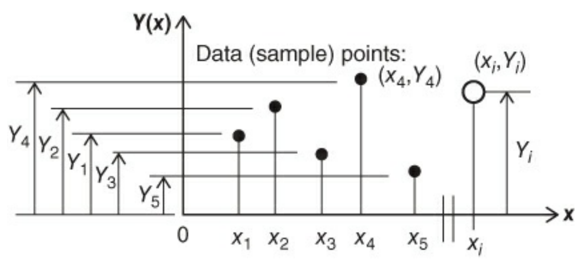

Figure 2.11 Coordinate system for polynomial function curve fitting.

Because the assumed polynomial function in Equation 2.7 is expected to pass through all the sample points with specified coordinates (x_i, Y_i) as indicated in the right portion of Figure 2.11, the following five simultaneous equations will be established for the five given sample points in Figure 2.10:

$$\begin{aligned}
Y_1 &= A_0 + A_1 x_1 + A_2 x_1^2 + A_3 x_1^3 + A_4 x_1^4 \\
Y_2 &= A_0 + A_1 x_2 + A_2 x_2^2 + A_3 x_2^3 + A_4 x_2^4 \\
Y_3 &= A_0 + A_1 x_3 + A_2 x_3^2 + A_3 x_3^3 + A_4 x_3^4 \\
Y_4 &= A_0 + A_1 x_4 + A_2 x_4^2 + A_3 x_4^3 + A_4 x_4^4 \\
Y_5 &= A_0 + A_1 x_5 + A_2 x_5^2 + A_3 x_5^3 + A_5 x_5^4
\end{aligned}$$

The five unknown coefficients, $A_0, A_1, \ldots, A_4$ can be determined by solving these five simultaneous equations.

Example 2.1

Derive a polynomial function that will pass the following three (3) data (sample) points: (1, 1.943), (2.75, 7.886), and (5, 1.738).

Solution:

We first note that the total number of sample point is 3, i.e., $N = 3$, giving 2 as the highest order of the assumed polynomial function: $n = N - 1 = 3 - 1 = 2$.

We assume a polynomial function of order of 2 in the form

$$Y(x) = A_0 + A_1 x + A_2 x^2 \qquad \text{a}$$

The assumed polynomial function in Equation (a) will fit (i.e., pass through) the three given data (sample) points with given coordinates:

$$Y_1 = 1.943 \text{ at } x_1 = 1, \qquad Y_2 = 7.886 \text{ at } x_2 = 2.75, \qquad Y_3 = 1.738 \text{ at } x_3 = 5$$

Consequently, by substituting the coordinates of the sample points into Equation (a), we will obtain three simultaneous equations:

$$\text{For } i = 1: \quad A_0 + A_1(1) + A_2(1)^2 = 1.943 \qquad \text{b1}$$

$$\text{For } i = 2: \quad A_0 + A_1(2.75) + A_2(2.75)^2 = 7.886 \qquad \text{b2}$$

$$\text{For } i = 3: \quad A_0 + A_1(5) + A_2(5)^2 = 1.738 \qquad \text{b3}$$

Solving the simultaneous equations in Equations (b1), (b2) and (b3), we obtain the values of the three unknown coefficients as

$$A_0 = -5.6663; \qquad A_1 = 9.1414; \qquad A_2 = -1.5321$$

Hence the polynomial function that will pass through the three given data points takes the form:

$$Y(x) = -1.5321x^2 + 9.1414x - 5.6663 \qquad \text{c}$$

The function in Equation (c) is shown graphically in Figure 2.12.

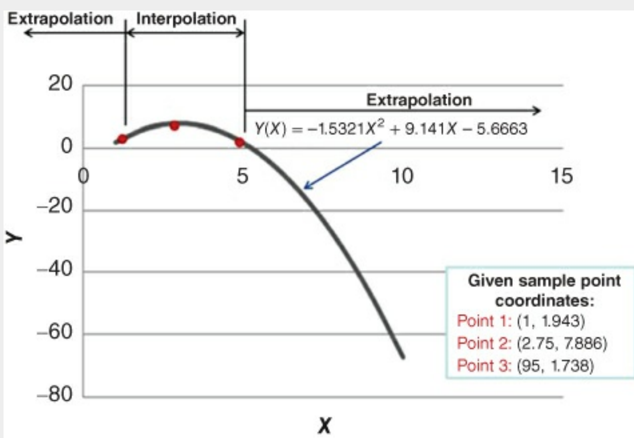

Figure 2.12 The fit of the derived function to the three given data (sample) points.

Having obtained this continuous function that includes the three data points, we may use it for interpolating the value of the function between the limits of the sample points, and also for extrapolating for the values outside the limits of the sample range, as illustrated in Figure 2.12.

2.2.5 Derivative

The values of most engineering quantities vary with *position* in the space, defined by the spatial variables, or/and with *time*, the temporal variable. For instance, the pressure in a moving fluid varies with position, and it also often varies with time if conditions change.

In most cases, variations of the value of functions occur *continuously* within the ranges of the associated variables, as illustrated in Figures 2.6 and 2.9.

2.2.5.1 The Physical Meaning of Derivatives

We define a derivative to be the *rate* of change of the value of a continuous function with respect to one of its associated variables.

Take for example the situation of a pressure chamber used in fabricating a product as illustrated in Figure 2.13. The process requires the product to be subjected to hydrostatic pressure. The chamber can be pressurized by regulating the pumping power and the pressure relief valve of the chamber. The rate of change of the pressure in the chamber varies as illustrated in Figure 2.14 in which the pressure at the closed circle points are the nine measured pressure readings at specific instances from the meter attached to the chamber. The rate of change of the pressure, P of the fluid at time t is expressed mathematically as $(\Delta P/\Delta t)$.

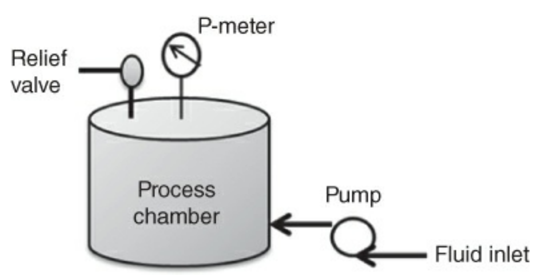

Figure 2.13 A pressurized process chamber.

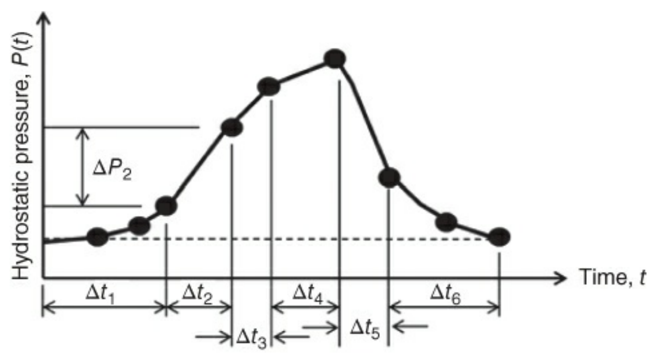

Figure 2.14 Variation of pressure in the process chamber.

We may observe from the measured pressure in the chamber as shown in Figure 2.14 that the rate of pressure variations represented by $\Delta P/\Delta t$ varies at different time periods Δt at which the measurements of the pressures in the process chamber were taken during the fabrication process. For example, the rate of pressure variation in the period designated Δt_2 is $\Delta P_2/\Delta t_2$ as shown in the figure. The rate of change of the function P in other periods Δt_i can be obtained using similar expressions, generalized as

$$\text{Rate of change of the value of the function } P = \frac{\Delta P_i}{\Delta t_i} \qquad \textbf{2.8}$$

with $i = 1, 2, 3, \ldots, 6$ in Figure 2.14.

At times the value of a function that represents a physical situation may vary continuously with the continuous variation of the associate variable, such as in the case of the pressure in a pressurized process chamber in Figure 2.13. One may readily conceive that the continuous variation of the pressure P in the chamber with time t cannot be represented by the connection of straight lines of all the measured pressures. The real situation would be more like that represented by the dashed curve in Figure 2.15. The rate of change of the function P can no longer be evaluated by Equation 2.8 because the value of P is changing at all times, not in distinct increments over a finite increment of the variable Δt as was represented in Figure 2.14. A different mathematical expression of $P(t)$ derived by a curve fitting technique such as presented in Section 2.2.4 is thus required.

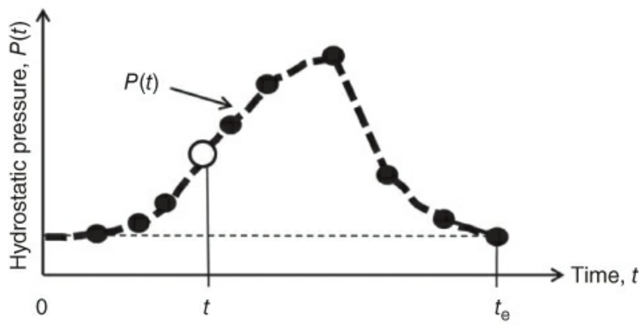

Figure 2.15 Pressure variations in the process chamber in Figure 2.13.

2.2.5.2 Mathematical Expression of Derivatives

Derivatives are used to evaluate the continuous change of the value of a function with infinitesimally small increments in the associate variables. The increments of the variable are so small that we may use the "approximation" of letting $\Delta t \to 0$ in our mathematical formulations.

In Figure 2.16 we illustrate the function y with its values "continuously" varying with an independent variable x; expressed mathematically, $y = f(x)$.

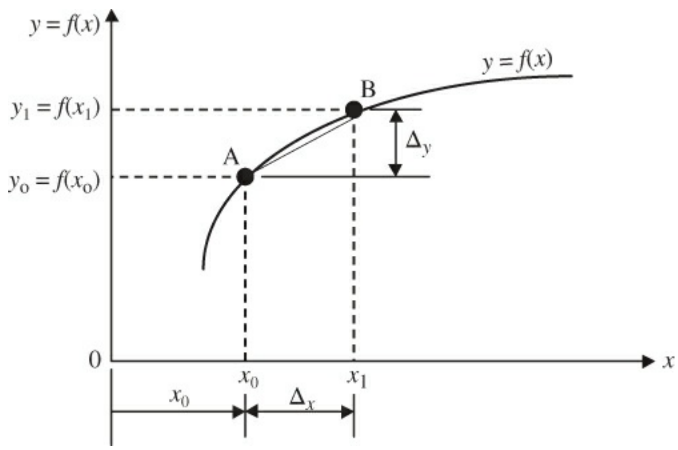

Figure 2.16 Continuous variation of a continuous function.

Let us pick two arbitrary points A and B on the curve that graphically represents the function $y = f(x)$. The function's values at these two points are $y_0 = f(x_0)$ at point A and $y_1 = f(x_1)$ at point B, where x_0 and x_1 are the corresponding values of variables associated with function values y_0 and y_1, respectively.

Let Δx = the increment of the variable x, then the corresponding change of the function values associated with this increment Δx is $\Delta y = y_1 - y_0 = f(x_1) - f(x_0)$. But we also have the relationship $x_1 = x_0 + \Delta x$ where Δx is the increment of the variable x as shown in Figure 2.16, which leads to the expression for the corresponding change of the value of the function:

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

The rate of change of $f(x)$ with respect to x is expressed as

$$\frac{\text{Change of value of function } f(x)}{\text{Change of value of } x\text{-coordinate}} = \frac{\Delta y}{\Delta x}$$
$$= \text{average rate of change within increment } \Delta x$$

This expression can be interpreted as the rate of change of the value of the function between point A and B in Figure 2.16; alternatively, it is the *slope of the curve* represented by $y = f(x)$ between point A and B as illustrated in Figure 2.16.

However, it must be realized Figure 2.16 that the average rate change between points A and B is not equal to the exact rate change at intermediate points between the two bounds A and B if Δx is *large*. The discrepancy becomes smaller with smaller Δx.

A more precise expression for the rate change of function $f(x)$ between x_0 and x_1 is obtained by reducing the size of the increment Δx. In other words, the rate of change of $y = f(x)$ can be represented accurate only if Δx is very small. The increment Δx is made so small that $\Delta x \to 0$. This, in reality, means the variation of the function with respective to the variation of x is *continuous*. Mathematically, it can be expressed as

$$\lim_{\delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\delta x\to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \left.\frac{dy(x)}{dx}\right|_{x=x_0} = \text{the derivative}$$

In general, the derivative of function $y = f(x)$ with respect to the variable x can be expressed in the following ways:

$$\frac{dy}{dx} = \frac{dy(x)}{dx} = \frac{df(x)}{dx}$$
$$= f'(x) = y'(x)$$
$$= \lim_{\delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{2.9}$$

Engineers should realize that not all functions, as represented by $y(x)$ in Equation (2.9), are differentiable. A function is differentiable only if the derivative exists.

Examples of derivatives of functions of various kinds are available in the literature (Ayres, 1964; Spiegel, 1963).

2.2.5.3 Orders of Derivatives

A function, $y(x)$, may be differentiated with its variable x more than once to give various “orders” of derivatives.

The derivative $dy(x)/dx$ in Equation (2.9) is referred to as the first-order derivative of the function $y(x)$, and may be itself a function or a constant.

The derivative of the derivative of the first order ($dy(x)/dx$)is called the second-order derivative. Mathematically, it is expressed as

$$\frac{d^2y(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy(x)}{dx}\right) = \text{the second-order derivative of function } y(x)$$
$$= \text{the rate of change of the first-order derivative}$$

There are also higher-order derivatives for some functions, such as the third- and fourth-order derivatives are shown below:

$$\frac{d^3y(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y(x)}{dx^2}\right) = \text{the third-order derivative of function } y(x)$$

$$\frac{d^4y(x)}{dx^4} = \frac{d}{dx}\left(\frac{d^3y(x)}{dx^3}\right) = \text{the fourth-order derivative of function } y(x)$$

Engineering analyses, especially mechanical and civil engineering analyses, rarely involve derivatives of higher order than 4.

2.2.5.4 Higher-order Derivatives in Engineering Analyses

Derivatives of a function of different orders have different physical meanings in engineering analysis. Figure 2.17 illustrates a beam in a deflected state due to applied bending loads, with the deflected shape of the beam depicted by the dashed line.

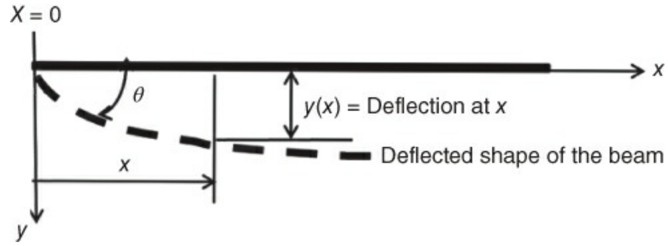

Figure 2.17 A deflected beam subject to bending load.

The following physical quantities required in beam stress analysis are given by derivatives of various orders of the beam deflection at location x, expressed as $y(x)$:

$$\frac{dy(x)}{dx} = \theta \text{ is the slope of the deflection curve of the beam at location } x \qquad \textbf{2.10a}$$

$$EI\left.\frac{d^2y(x)}{dx^2}\right|_x = M(x) \text{ is the bending moment in the beam at } x \qquad \textbf{2.10b}$$

where E and I are the Young's modulus of the beam material and the moment of inertia of the beam cross-section respectively.

$$C\left.\frac{d^3y(x)}{dx^3}\right|_x = V(x) \text{ is the shear force in the beam at } x \ \ (C \text{ is a constant}) \qquad \textbf{2.10c}$$

The deflection $y(x)$ of a beam of constant cross-section subjected to a distributed load $Q(x)$ can be obtained by solving the differential equation (Equation 2.11):

$$\frac{d^4y(x)}{dx^4} = \frac{Q(x)}{EI} \qquad \textbf{2.11}$$

We will illustrate solving for the deflection and the induced stresses of beams subjected to complex loading and end conditions using Equations 2.10 and Equation 2.11 in a later part of this chapter.

2.2.5.5 The Partial Derivatives

There are engineering problems that involve more than one independent variable in the analysis. The value of the function varies independently with each of the associated independent variables. Figure 2.18 illustrates such a case: that the cantilever beam shown in Figure 2.18a is subjected to a moving load at a variable velocity $v(t)$, where t is the time. Clearly, the load on the beam varies not only with the position x but also with the time t, as expressed in Figure 2.18b.

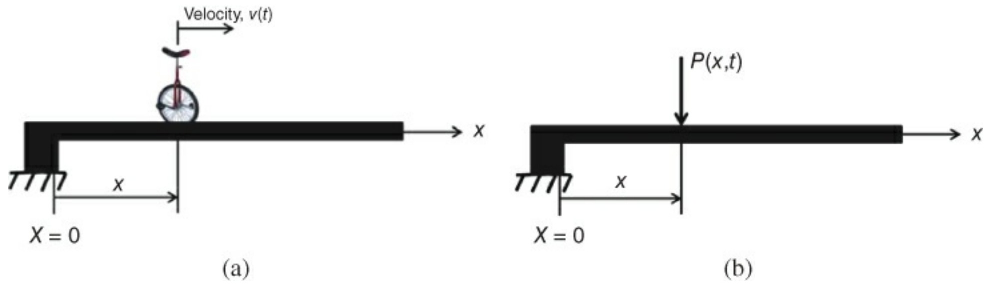

Figure 2.18 A cantilever beam subject to a moving load. (a) Moving load on the beam. (b) The loading function of the beam.

The derivatives of the loading function P given in Figure 2.18b are expressed in terms of each of the two independent variables x and t, but not with both variables included in one derivative. The rate of change of the loading function P needs to be evaluated separately with respect to each of these two variables. We have then two derivatives representing the rate change of the function $P(x,t)$:

$$\frac{\partial P(x,t)}{\partial x} \quad \text{and} \quad \frac{\partial P(x,t)}{\partial t} \text{ for the first-order partial derivatives}$$

and

$$\frac{\partial^2 P(x,t)}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 P(x,t)}{\partial t^2} \text{ for the second-order partial derivatives}$$

Note that the differential operators $\partial/\partial x$ and $\partial/\partial t$ are used in the above derivatives to indicate these are derivatives of a function with respect to the "partial" not the full rate of change of the function values with both variables associated with the function.

2.2.6 Integration

2.2.6.1 The Concept of Integration

Contrary to differentiation, to obtain the derivatives of functions, which account for the rate of change of the value of a function within selected infinitesimally small increments of the associated variable, integration sums the areas of infinitesimally small elements of area under the graph of the curve $y = f(x)$ with infinitesimally small increments of the associated variable x. Thus, the integral accounts for the total area bounded by the curve defined by the function at $x = a$ and $x = b$ with infinitesimally small increments of variable x, or $\Delta x \to 0$, as illustrated in Figure 2.19, where a and b are the limits of the integration. There are two types of integrals: definite integrals and indefinite integrals. Definite integrals involve a definite range of the associated variable in the determination of the area bounded by the curve $y(x)$ between these limits; indefinite integrals have no specified range of variable in the evaluations.

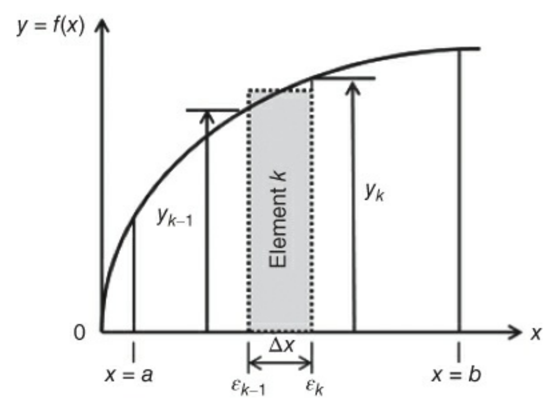

Figure 2.19 Illustration of the concept of integration.

2.2.6.2 Mathematical Expression of Integrals

Figure 2.20a represents a measured continuous increase of pressure in the process chamber illustrated in Figure 2.13. We denote the pressure increase with time t as $P(t)$. We readily evaluate the rate of the pressure increase at time t_k to be $dP(t)/dt|_{t=t_k}$ according to Equation 2.9. The formula for the area bounded by the curve of function $P(t)$ can be derived by referring to Figure 2.20b.

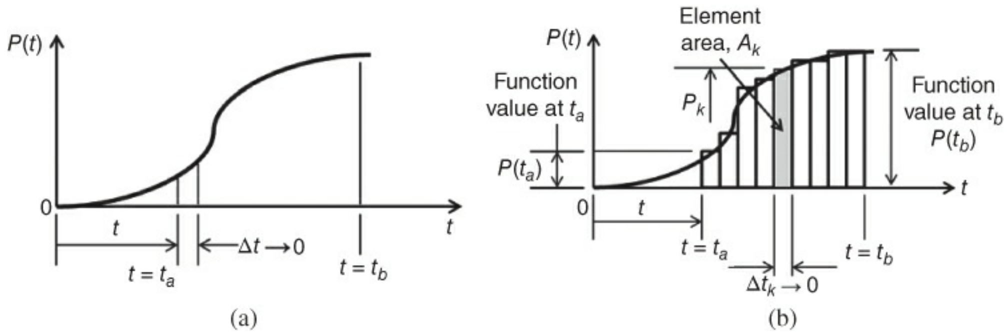

Figure 2.20 Area bounded by a continuous function $P(t)$. (a) A continuous function $P(t)$. (b) The elements of area bounded by $P(t)$.

Referring to the general case of small element k bounded by function $y = f(x)$ in Figure 2.19, the area of the element is $A_k \approx y_{i-1}\Delta x \approx y_k \Delta x$ with $y_{i-1} \approx y_k$ because the two are located very closely together as $\Delta x \to 0$. The same may be applied to the formulation of element area A_k in Figure 2.20b as $A_k \approx P_k \Delta t_k$. The total area bounded by the curve of the function $P(t)$ between values of the variable of t_a and t_b is the sum of all the element areas:

$$A = \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} P_k \, \Delta t_k \quad \mathbf{2.12}$$

We can see from Figure 2.20b that the rectangular-shaped elements shown in Figures 2.19 and 2.20 will more accurately represent the curve represented by the functions $y = f(x)$ in Figure 2.19 and $P(t)$ in Figure 2.20b as the corresponding increments Δx and Δt_k become small; and the true representation of the real continuous functions will be obtained as the increments become infinitesimally small, i.e., as $\Delta x \to 0$ in Figure 2.19 and $\Delta t_k \to 0$ in Figure 2.20b. Consequently, we have the following expression for the definite integral for continuous functions $f(x)$ (as in Figure 2.19):

$$A = \lim_{\delta n \to 0} \sum_{k=1}^{n} y_k \, \Delta x_k = \int_a^b f(x)\, dx \quad \mathbf{2.13}$$

Similarly, the integral for the situation depicted in Figure 2.20 can be expressed as

$$A = \int_{t_a}^{t_b} P(t)\, dt$$

The results of integration of many different forms of functions can be found in references such as Zwillinger (2003).

If we let $F(x)$ be a function whose derivative, $F'(x) = f(x)$, then the function $F(x)$ is an *"Anti-derivative" or "indefinite integral"* of *f(x)*. Mathematically, this relationship is expressed as

$$F(x) = \int F'(x)\, dx = \int f(x)\, dx$$

The function $f(x)$ in the integral is called the *integrand*.

Integration of functions has many applications in engineering analysis. It is used to evaluate areas and volumes enclosed by curves represented by functions. Integration is also used to find geometric quantities such as centroids of plane areas and centers of gravity of solids, as well as moment of inertias for cross-sections of beams and mass moment of inertia of solids involved in curvilinear motion. These geometric properties are frequently involved in many computer-aided design analyses.

2.3 Applications of Integrals

2.3.1 Plane Area by Integration

In Section 2.2.6, we derived the formula for integrals in Equation 2.13 for the area bounded by the curve represented by the function $f(x)$. We will use the same formula for determining the plane areas that can be described by continuous functions.

Example 2.2

Find the area of the right triangle in Figure 2.21 using an integration method.

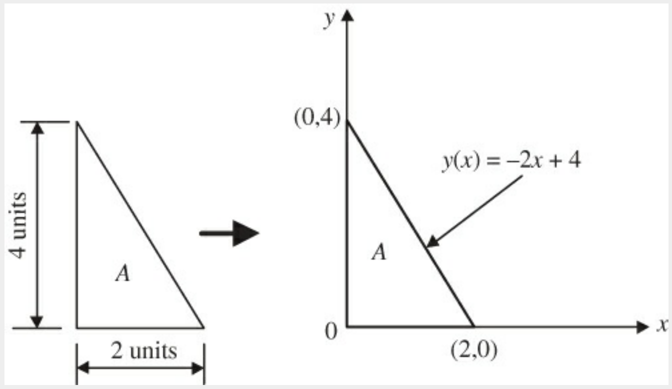

Figure 2.21 Plane area of a right triangle.

Solution:

In this rather special case, we may take advantage of the right angle between the two shorter sides of the triangle by setting the x–y coordinates. This would leave the inclined side of the triangle (the hypotenuse) to be the "curve" under which the area is to be evaluated as shown in Figure 2.21. Thus the "curve" is actually a straight line and the mathematical expression to describe this straight line and thus the "curve" is $y(x) = -2x + 4$. Consequently, the area that is enclosed by this straight line is

$$A = \int_0^2 y(x)\,dx = \int_0^2 (-2x+4)\,dx = (-x^2+4x)|_0^2 = 4$$

by using Equation 2.13.

Example 2.3

Determine the plane area of a quarter-circular plate with dimensions as shown in Figure 2.22.

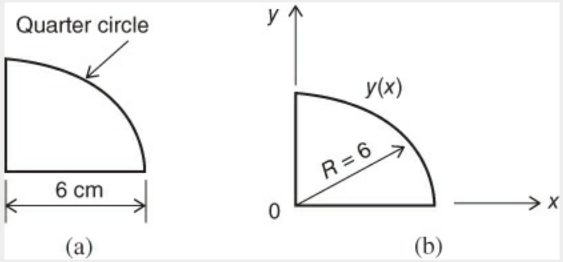

Figure 2.22 Plane area of a quarter circular plate. (a) Plate geometry. (b) The plate in an *x*–*y* coordinate system.

Solution:

Since the curved edge of the plate shown in Figure 2.22b fits in a circle, and the equation of a circle is $x^2 + y^2 = R^2 = (6)^2 = 36$, from which we may establish the following functions:

$$x(y) = \sqrt{36 - y^2} \tag{2.14a}$$

$$y(x) = \sqrt{36 - x^2} \tag{2.14b}$$

The area A under the arc between x = 0 and x = 6 in Figure 2.22b can thus be determined by using formula in Equation 2.13 as

$$\begin{aligned} A &= \int_0^6 y(x)\,dx = \int_0^6 \sqrt{36 - x^2}\,dx \\ &= \frac{1}{2}\left(x\sqrt{36 - x^2} + 36\sin^{-1}\frac{x}{6}\right)\Bigg|_0^6 \\ &= 9\pi \quad \text{or} \quad 28.26\ \text{cm}^2 \end{aligned}$$

Example 2.4

Find the area of the plate in Figure 2.23 with a curved edge that fits a cosine function.

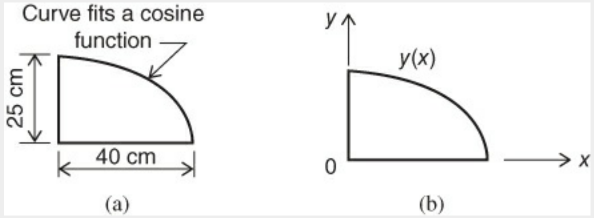

Figure 2.23 Plate with a curved edge. (a) Plate geometry. (b) The plate in an *x*–*y* coordinate system.

Solution:

The curved edge of the plate in Figure 2.23b fits a function $y(x) = 25\cos[(\pi/80)x]$. Thus the area of the plate can be obtained using the integral shown in Equation (2.13):

$$A = \int_0^{40} \left(25\cos\frac{\pi}{80}x\right) dx = 636.94 \ \text{cm}^2$$

Example 2.5

Find the area of a plate with a curved edge that fits an ellipse as shown in Figure 2.24a.

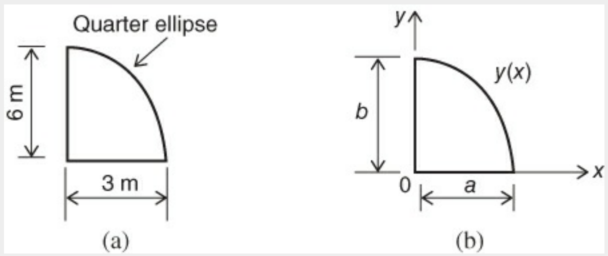

Figure 2.24 Plate with a curved edge that fits an ellipse. (a) Plate geometry. (b) The plate in an *x*–*y* coordinate system.

Solution:

The function $y(x)$ that represents an ellipse may be derived from the equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

With a = 3 m and b = 6 m, we will have the function $y(x) = 2\sqrt{9 - x^2}$. Thus, by substituting this function into Equation (2.13), we obtain the area of the plate as

$$A = \int_0^3 (2\sqrt{9 - x^2})\,dx = \frac{9\pi}{2} = 14.13 \ \ \text{m}^2$$

Example 2.6

Determine the area of a plate with the geometry and overall dimensions shown in Figure 2.25a.

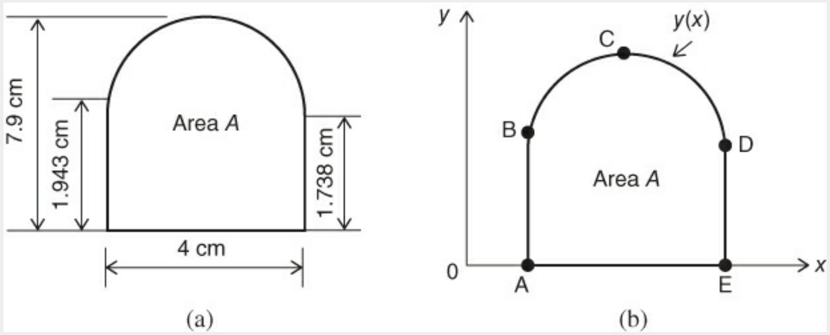

Figure 2.25 Plate with curved edge by a designer. (a) Geometry of the plate. (b) The plate in an x–y coordinate system.

Solution:

The curved edge of the plate as shown in Figure 2.25 was the work of a designer. It does not fit to any common function as were the cases in Examples 2.2 to 2.5. Consequently, one needs to use a curve fitting technique to develop a continuous function to describe this curved edge.

We may place the plate in an x–y coordinate system as shown in Figure 2.25b, in which five points, A, B, C, D, and E, are used to define the geometry of the plate with the coordinates of each point being A(1,0), B(1, 1.943), C(2.75, 7.886), D(5,1.738), E(5,0).

We may use the polynomial curve fitting technique described in Section 2.2.4 with these measured data points to produce a polynomial function with the three data points B, C, and D similarly to what was done in Example 2.1.This gives the function that fits the curved portion of the plate as

$$y(x) = -1.5321x^2 + 9.1414x - 5.6663$$

The area of the plate can thus be determined from the integral

$$\begin{aligned} A_1 &= \int_1^5 y(x)\,dx \\ &= \int_1^5 (-1.5321x^2 + 9.1414x - 5.6663)\,dx \\ &= 23.7\ \ \text{cm}^2 \end{aligned}$$

The area of the other part, the trapezoid defined by AEDB is

$$A_2 = \frac{(1.738 + 1.943)4}{2} = 7.362\ \ \text{cm}^2$$

Hence the total plane area of the plate is $A = A_1 + A_2 = 31.062\ \text{cm}^2$.

2.3.1.1 Plane Area Bounded by Two Curves

Integration can also be used to determine areas enclosed by two functions $f(x)$ and $g(x)$ as illustrated in Figure 2.26.

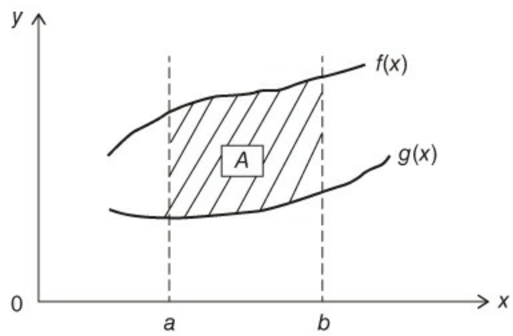

Figure 2.26 Plane area between two curves.

The area defined by the two functions in Figure 2.26 between the limits $x = a$ and $x = b$ is

$$\begin{aligned} A &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx \end{aligned} \tag{2.15}$$

Example 2.7

Determine the plane area between a half ellipse and a half circle as shown in Figure 2.27a.

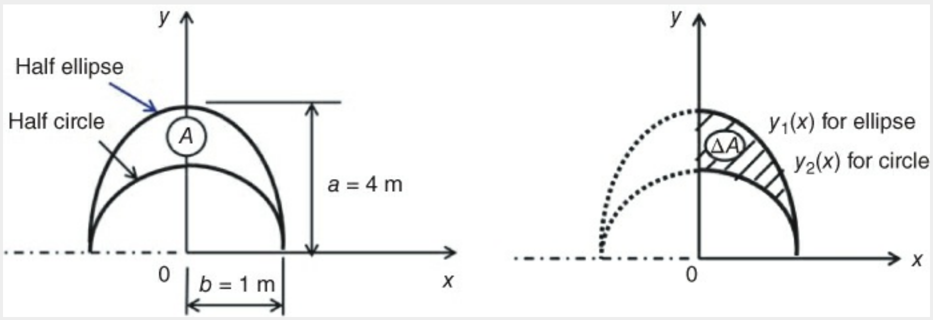

Figure 2.27 Area between a half ellipse and a half circle. (a) Plane area defined by two curves. (b) Symmetry of the geometry about the *y*-axis.

Solution:

Making use of Figure 2.26 and Equation 2.15, the area between the two curves in Figure 2.27a may be determined defining these two curves in the *x*–*y* coordinate systems and using the symmetry of the plane about the *y*-axis as shown in Figure 2.27b.

The function $y_1(x)$ in Figure 2.27b may be expressed using the equation of ellipse:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

from which we obtain

$$y(x) = \frac{a}{b}\sqrt{b^2 - x^2}$$

With a = 4 m and b = 1 m from Figure 2.27a, we have the function $y_1(x) = 4\sqrt{1 - x^2}$.

The other function, $y_2(x)$ may be derived from the equation of circle: $x^2 + y^2 = r^2$, from which we obtain $y(x) = \sqrt{r^2 - x^2}$. With the radius of the circle being r = 1 m, we have the function $y_2(x) = \sqrt{1 - x^2}$.

By substituting both $y_1(x)$ and $y_2(x)$ into Equation 2.15, we get the area between these two functions:

$$\begin{aligned}\Delta A &= \int_0^1 [y_1(x) - y_2(x)]\,dx \\ &= \int_0^1 4\sqrt{1 - x^2}\,dx - \int_0^1 \sqrt{1 - x^2}\,dx \\ &= 3\int_0^1 \sqrt{1 - x^2}\,dx = 4.71 \ \ \mathrm{m}^2\end{aligned}$$

which leads to a total area between the two functions of $2 \times \Delta A = 9.41\ \mathrm{m}^2$.

2.3.2 Volumes of Solids of Revolution

This type of solid has a geometry that is symmetrical about one axis (“the axis of symmetry” in Figure

2.28). It is common in mechanical engineering applications. Typical solids of revolution include cylinders, cones, conical frustums, nozzles, etc. These solids are made up by revolving a plane area about a line, which is often called the *axis of rotation*.

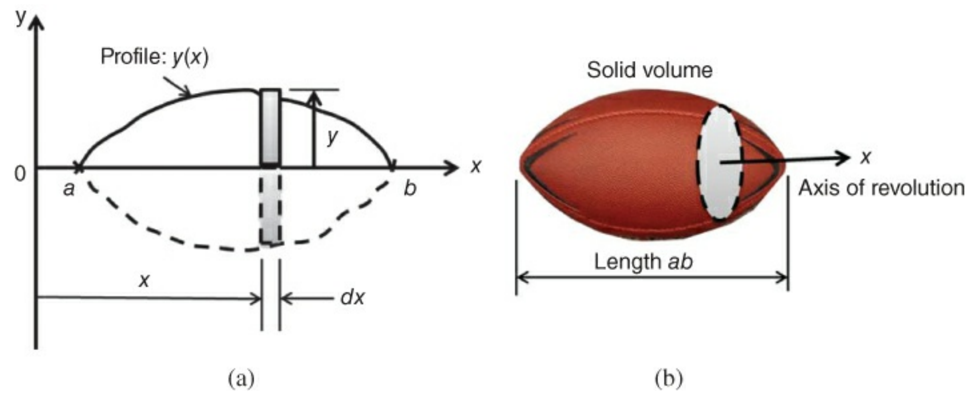

Figure 2.28 Solid of revolution. (a) Exterior profile of the solid. (b) The same solid with axis of revolution.

Often, engineers need to find the volume of a solid of revolution in order to determine the mass or weight of the solid in many design analyses. The volumes of such solids can be obtained using the special definite integrals described here.

Figure 2.28a indicates a function $y(x)$ that represents the profile of the solid of revolution shown in Figure 2.28b. The gray element of volume shown in Figure 2.28a corresponds in position to the cross-section of the solid normal to the axis of revolution that is shown in Figure 2.28b.

The volume of the solid in Figure 2.28b may be obtained by summing up the volumes of small elements in Figure 2.28a. These elements may be viewed as thin "disks" with radius $y(x)$ and thickness Δx. This is expressed mathematically as

$$\Delta V_i = (A_i)\,dx = (\pi r^2)\Delta x$$
$$= \pi[y(x)]^2 \Delta x$$

where A_i and r are the cross-sectional area and the radius of the "thin" disk element, respectively.

The volume of the entire solid is the sum of the volumes of all these "thin" disk elements along the axis of revolution, or

$$V = \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} [\pi(y(x))^2]\,\Delta x$$

in which i designates the disk element number and n is the total number of the disk elements in the subdivided solid.

Because the function $y(x)$ that represents the profile of the solid is a continuous function as shown in Figure 2.28a, we may express the volume of entire solid with the condition that $\Delta x \to dx$ with $dx \to 0$. The summation sign in the above expression for the volume V is thus replaced by the integral with the form

$$V = \int_a^b [\pi y(x)^2]\,dx$$
$$= \pi \int_a^b [y(x)^2]\,dx \qquad \textbf{2.16}$$

The volume of a solid of revolution about the y-axis, such as shown in Figure 2.29, can be determined in a similar manner.

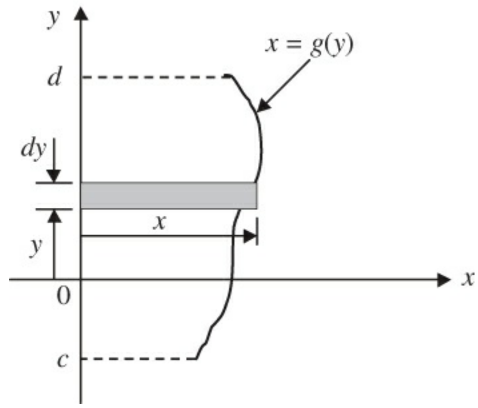

Figure 2.29 Solid volume of revolution about the *y*-axis.

The area of the shaded element can be expressed as $dA = \pi x^2$ and the volume of revolution of the shaded element is $dv = \pi x^2\, dy$. The total volume of the solid of revolution about the *y*-axis can be expressed as

$$V = \pi \int_c^d x^2 \ dy = \pi \int_c^d [g(y)]^2 \ dy \qquad \text{2.17}$$

Example 2.8

Determine the volume of the right cone shown in Figure 2.30 by using an integration method.

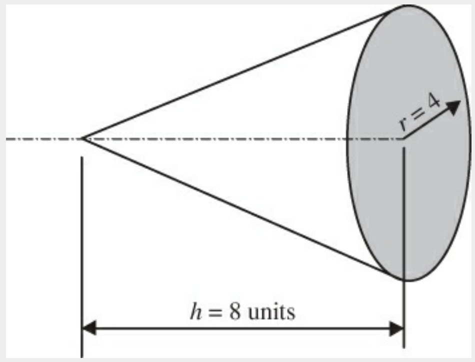

Figure 2.30 A right solid cone.

Solution:

The right solid cone in Figure 2.30 can be shaped by the rotation of a straight line described by a function $y(x) = 0.5x$ about the x-axis as illustrated in Figure 2.31.

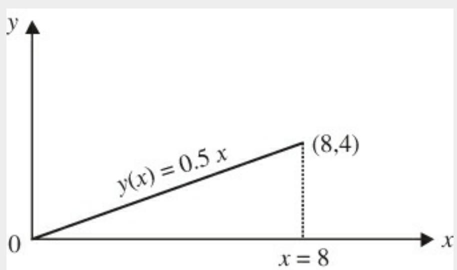

Figure 2.31 Function describing a cone.

The volume of the solid of revolution of the function $y(x)$ about the x-axis is determined using Equation (2.16):

$$V = \pi \int_0^8 [y(x)]^2\,dx = \pi \int_0^8 (0.5x)^2\,dx = 42.67\pi$$

One may find from engineering handbooks that the volume of a right cone is equal to

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(4)^2 \times 8 = 42.67\pi$$

which is identical to the result determined using the integration method.

Example 2.9

Find the volume generated by revolving the first-quadrant area bounded by the parabola $y^2 = 8x$ and its latus rectum ($x = 2$) about the x-axis as illustrated in Figure 2.32.

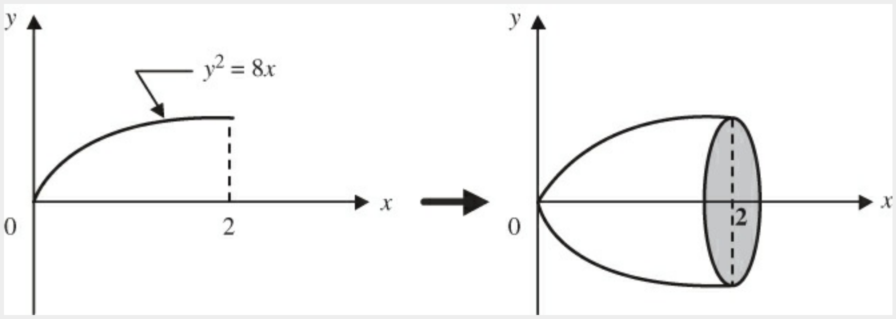

Figure 2.32 A solid volume of revolution of a parabolic curve.

Solution:

Using Equation 2.16 we will find the volume of revolution V to be

$$V = \pi \int_a^b [f(x)]^2\, dx = \pi \int_0^2 (8x)\, dx = 4\pi x^2 \Big|_0^2 = 16\pi$$

Example 2.10

Use the integration method to determine the volume of a wine bottle as shown in Figure 2.33a. The dimensions of the bottle are shown in Figure 2.33b.

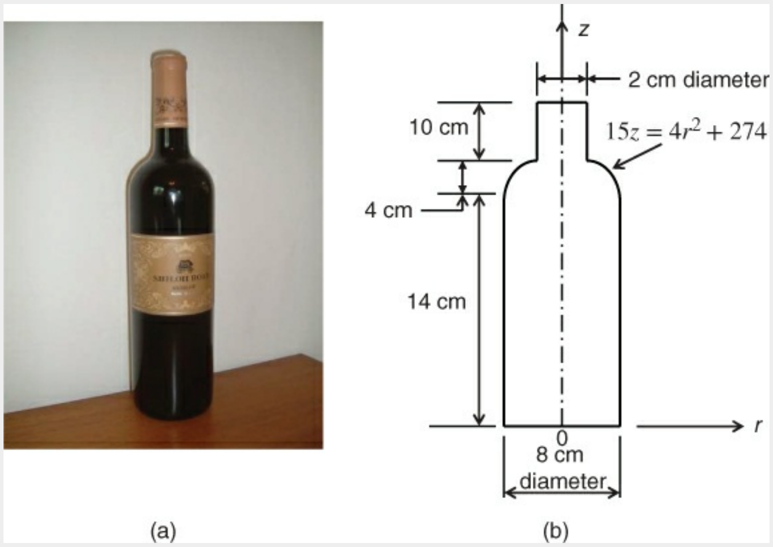

Figure 2.33 A wine bottle. (a) The physical bottle. (b) The profile and dimensions of the bottle.

Solution:

Figure 2.33b shows that the wine bottle is made up of three sections: two straight sections and one with curved cross-section. We may readily determine the volumes of the two straight sections with the expression $V = [(\pi d^2)/4]L$, in which d and L are the respective diameters and lengths of the straight sections of the wine bottle. The volume of the curved part of the bottle requires the use of the integral expression in Equation 2.17.

Let us use the coordinate system (r,z) established as in Figure 2.29, with the radial coordinate r coinciding with the x-coordinate in Figure 2.29 and the axial coordinate z in place of the y-axis in the same figure. Letting V_1 = the volume of the top straight-sided section and V_2 = the volume of the bottom straight sided section:

$$V_1 = \frac{\pi}{4}d_1^2 L_1 = 0.785(2)^2 \times 10 = 31.4 \text{ cm}^3$$
$$V_2 = \frac{\pi}{4}d_2^2 L_2 = 0.785(8)^2 \times 14 = 703.36 \text{ cm}^3$$

We will use the polynomial curve fitting technique described in Section 2.2.4 to fit the profile of the curved portion of the wine bottle with three measure diameters along the z-axis ,the axis of revolution, as shown in Figure 2.34.

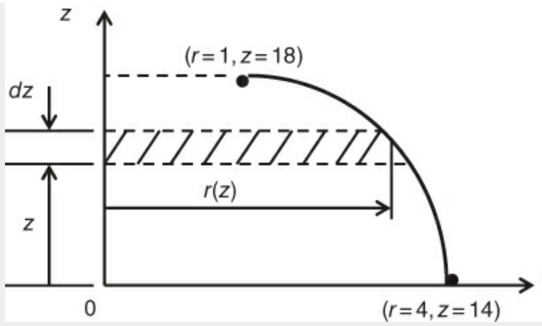

Figure 2.34 Volume of revolution of the curved section of the wine bottle.

Using the formulas derived in Section 2.2.4, we can determine the volume of this intermediate section from the cross-hatched volume element in Figure 2.34 to be

$$dV_3 = \pi[r(z)]^2\,dz$$

where the fitted curved profile of this section of the wine bottle can be represented by a polynomial function $15z = -4r^2 + 274$. Consequently, the total volume in this portion of the wine bottle under the above function is computed as

$$V_3 = \int_{14}^{18} \pi[r(z)]^2\,dz = \pi\int_{14}^{18} r^2\,dz = \pi\int_{14}^{18} \frac{274 - 15z}{4}\,dz = 106.76\ \text{cm}^3$$

The total volume of the wine bottle is then equal to the sum of the volumes of the three sections: $V_1 + V_2 + V_3 = 841.52\ \text{cm}^3$.

The total volume as computed using the integration method was larger than the 750 cm^3 shown in the label of the wine bottle. The discrepancy was mainly due to the discounting of the volume of the "punt" of the wine bottle in the above computation. Figure 2.35 shows the punt of a typical wine bottle.

Figure 2.35 The "punt" at the bottom of a wine bottle.

2.3.3 Centroids of Plane Areas

The centroid of a solid of plane geometry is the point that coincides with the center of gravity of the solid. In engineering analysis, we often consider that the mass or weight of machine components of plane geometry (e.g., a *gear* or *plate*) is "concentrated" at the centroid in a typical dynamic analysis of a rigid body of planar geometry in motion (Meriam and Kraige. 2007). Other examples of involving determining the location of centroids of solids of plane geometry include the moving couplers in mechanisms such as illustrated in Figure 2.36. The coupler that is attached to a 4-bar linkage is designed to have its tip A follow a prescribed trajectory with the rotating crank. The induced dynamic force in this oscillating coupler can be a major load to the linkage, and the location of the center of gravity of this coupler of solid plane geometry must be determined in the subsequent stress analysis for the entire mechanism. Figure 2.37 shows an assembly of a mechanism involving a cam and follower that is common in many mechanical systems. The cam, which is usually made in plane geometry, rotates about an axle. The rotary motion of the cam pushes the follower to move up and down according to the profile of the cam. In both of these systems—the coupler of a 4-bar linkage and the cam–follower assembly—rapid motion or rotation of the coupler and the cam is common in practice. The dynamic forces induced by these motions can be significant, and the location of the points of application of force is critical in the design analyses. Identification of the centroids of these planar solids is thus an important part of the design analysis.

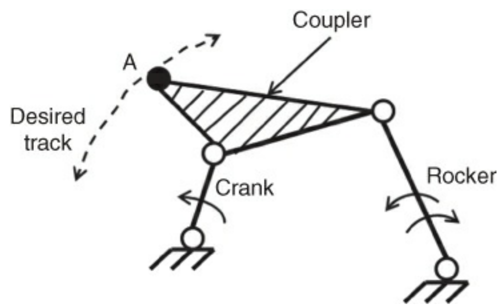

Figure 2.36 Mechanism with a coupler.

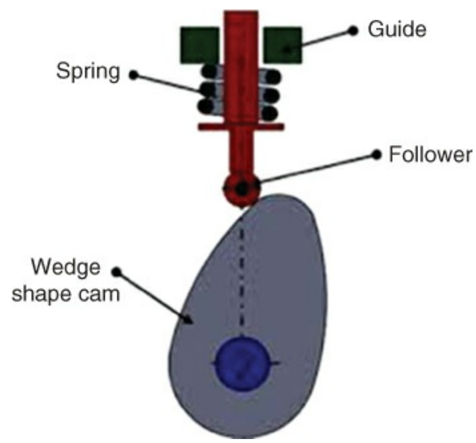

Figure 2.37 Cam with follower.

A solid of arbitrary plane geometry is shown in Figure 2.38, for which the centroid of the solid plane is located at the coordinates $(\bar{x}, \bar{y})$.

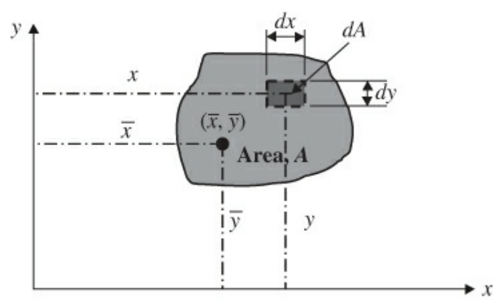

Figure 2.38 Centroid of a plane.

The area of a small element of the plane, such as the one shaded in dark gray in Figure 2.38, is expressed as $dA = dx\,dy$. We will define the area moment of the element area dA about the x-axis to be $dM_x = y(dA) = y\,dx\,dy$, and the area moment about the y-axis to be $dM_y = x\,dA = x\,dx\,dy$.

The total area moments can thus be obtained by summing the area moments over all area elements in the entire plane solid:

Area moment about the x-axis:

$$M_x = \int_A dM_x = \int_A y\,dA = \int_x \int_y y(dy\,dx) \qquad \textbf{2.18a}$$

Area moment about the y-axis:

$$M_y = \int_A dM_y = \int_A x\,dA = \int_y \int_x x(dx\,dy) \qquad \textbf{2.18b}$$

The coordinate of the centroid of the entire plane can thus be determined by the relations

$$\bar{x} = \frac{M_y}{A} \qquad \textbf{2.19a}$$

$$\bar{y} = \frac{M_x}{A} \qquad \textbf{2.19b}$$

where the area of the plane is

$$A = \int_A dA = \int_y \int_x dx\,dy \qquad \textbf{2.20}$$

2.3.3.1 Centroid of a Solid of Plane Geometry with Straight Edges

In the case of solids of plane geometry with at least one straight edge, such as the edge C–D of the plane ABCD shown in Figure 2.39, we may set the straight edge to coincide with one of the coordinates as illustrated in the same figure. The area moments for such case can be expressed as

$$M_x \int_{x_1}^{x_2} \left(\frac{1}{2}y\right) dA = \frac{1}{2}\int_{x_1}^{x_2} [y(x)]^2\,dx \qquad \textbf{2.21a}$$

$$M_y = \int_A x\,dA = \int_{x_1}^{x_2} xy(x)\,dx \qquad \textbf{2.21b}$$

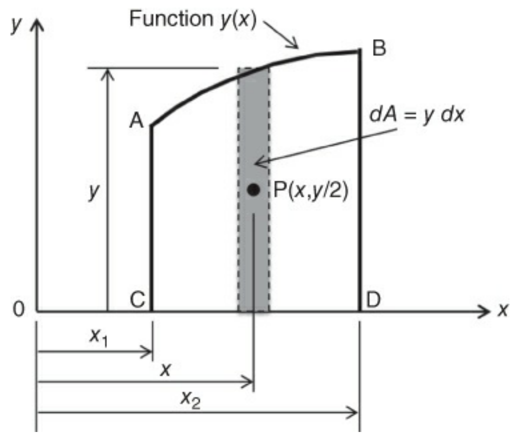

Figure 2.39 Solid of plane geometry with straight edges.

The coordinates of the centroid $(\bar{x}, \bar{y})$ of the plane ABCD can be obtained by using Equations 2.21a and 2.21b with the area obtained from Equation 2.20

Example 2.11

Determine the location of the centroid of a semicircle with radius a as illustrated in Figure 2.40:

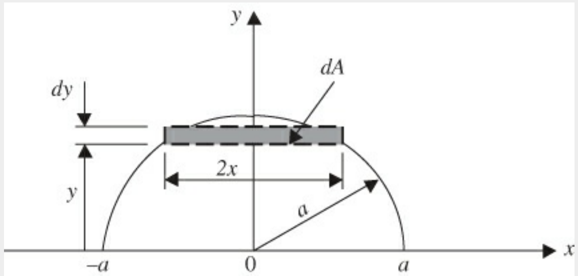

Figure 2.40 Centroid of a semicircular plate.

Solution:

The equation describing a circle of radius a is $y^2 + x^2 = a^2$ from analytical geometry, from which we have the functions

$$y(x) = \sqrt{a^2 - x^2} \quad \text{or} \quad x(y) = \sqrt{a^2 - y^2}$$

We realize by observation that there is no need to determine the x-coordinate of the centroid because of the symmetry of the geometry about the y-axis of the semicircle. This symmetry in geometry leads to $\bar{x} = 0$. However, the determination of the y-coordinate of the centroid (i.e., $\bar{y}$) requires the determination of the area moment M_x as indicated in Equation 2.19.

The plane area in Figure 2.40 shows the area of the element $dA = 2x\,dy$ and the area moment is $M_x = y\,dA$ from Equation 2.18a. Thus, the y-coordinate of the centroid is determined using Equation 2.21a to be

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{2\int xy\,dy}{2\int x\,dy} = \frac{\int_0^a y\sqrt{a^2 - y^2}\,dy}{\int_0^a \sqrt{a^2 - y^2}\,dy}$$

At this stage of the analysis, the engineer may either use an electronic calculator to perform the integrations in the above expression, or use mathematical handbooks such as the *CRC Standard Mathematical Tables and Formulae* (Zwillinger, 2003) to identify the solution. The latter approach will lead to the following for the location of the centroid along the y-coordinate:

$$\bar{y} = \left. \frac{-\frac{1}{3}\sqrt{(a^2 - y^2)^3}}{\frac{1}{2}\left[y\sqrt{a^2 - y^2} + a^2 \sin^{-1}\frac{y}{|a|}\right]} \right|_0^a = \frac{\frac{1}{3}a^3}{\frac{1}{2}a^2\left(\frac{\pi}{2}\right)} = \frac{4a}{3\pi}$$

2.3.3.2 Centroid of a Solid with Plane Geometry Defined by Multiple Functions

Cases arise of solids of plane geometry with one edge defined by a multiple functions, such as that illustrated in Figure 2.41. In Figure 2.41, the top edge of the plane is defined by three functions; $y_1(x)$, $y_2(x)$, and $y_3(x)$. One may divide the plane into three regions: A_1, A_2, and A_3. For the corresponding areas of these three subdivisions:

$(\bar{x}_1, \bar{y}_1)$ = the coordinates of the centroid in subdivision A_1

$(\bar{x}_2, \bar{y}_2)$ = the coordinates of the centroid in subdivision A_2

$(\bar{x}_3, \bar{y}_3)$ = the coordinates of the centroid in subdivision A_3

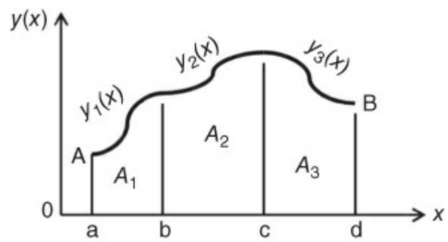

Figure 2.41 Solid of plane geometry with edges defined by multiple functions.

The centroid of the entire plane (ABda), denoted $(\bar{x}, \bar{y})$, may be obtained as the weighted average of those of the three subdivisions as follows:

$$\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3}{A_1 + A_2 + A_3} \quad \mathbf{2.22a}$$

$$\bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3}{A_1 + A_2 + A_3} \quad \mathbf{2.22b}$$

Example 2.12

Determine the location of the centroid in a coupler made of a triangular plate as illustrated in Figure 2.42a, with the dimensions shown in Figure 2.42b.

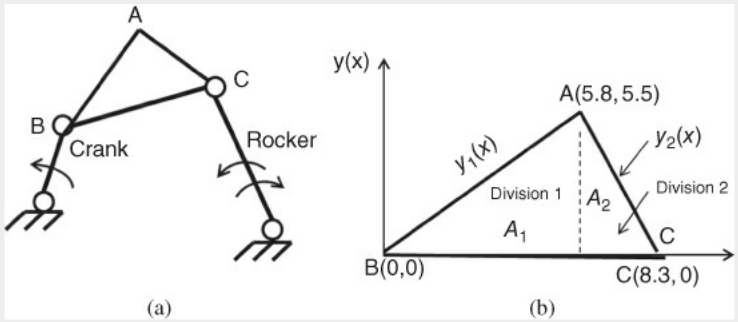

Figure 2.42 Four-bar linkage with a triangular coupler. (a) The coupler ABC. (b) Dimensions of the coupler.

Solution:

The plate in Figure 2.42a has three straight edges, so we may choose to have one of the edges BC to coincide with the *x*-axis as shown in Figure 2.42b. This leaves the two edges AB and AC to define the plate geometry with two distinct functions: $y_1(x)$ for edge AB and $y_2(x)$ for the other edge, in analogy with Figure 2.41.

We first derive the functions $y_1(x)$ and $y_2(x)$ using the coordinates of apexes A, B, and C in Figure 2.42b:

$$y_1(x) = 0.9483x \tag{a}$$

$$y_2(x) = -2.2x + 18.26 \tag{b}$$

We may compute the plane areas for the two divisions in Figure 2.42b, as well as the centroid of these two divisions using Equations 2.13, 2.19, and 2.21 as follows:

$$A_1 = \int_0^{5.8} y_1(x)\,dx = 15.95$$
$$M_{x1} = \frac{1}{2}\int_0^{5.8} [y_1(x)]^2\,dx = 29.24$$
$$M_{y1} = \int_0^{5.8} x y_1(x)\,dx = 61.6743$$

We thus have:

$$\bar{x}_1 = \frac{M_{y1}}{A_1} = 3.8667 \quad \text{and} \quad \bar{y}_1 = \frac{M_{x1}}{A_1} = 1.8332$$

We may proceed to division 2, following the same procedure as for division 1, and obtain the following:

$$A_2 = \int_{5.8}^{8.3} y_2(x)\,dx = 6.875$$

$$M_{x2} = \frac{1}{2}\int_{5.8}^{8.3} [y_2(x)]^2\,dx = 12.6822$$

$$M_{y2} = \int_{5.8}^{8.3} xy_2(x)\,dx = 45.6042$$

with

$$\bar{x}_2 = \frac{M_{y2}}{A_2} = 6.6333 \quad \text{and} \quad \bar{y}_2 = \frac{M_{x2}}{A_2} = 1.8374$$

The coordinates of the centroid for the entire region can thus be obtained from Equations 2.22a,b to be:

$$\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2}{A_1 + A_2} = 4.7 \quad \text{and} \quad 4\bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2}{A_1 + A_2} = 1.8345$$

2.3.4 Average Value of Continuous Functions

Often, engineers need to determine the average value of a function that represents a certain physical phenomenon or quantity. It is not a problem for functions with discrete values. For instance, the average load on a beam in Figure 2.5 can be easily determined by adding the four discrete values of the weights of people standing on the beam and then dividing the total weight by a factor of 4. The answer to the same question for the average value of continuous functions such as those illustrated in Figures 2.6 and 2.15, however, cannot be found by calculating the arithmetic average. Integration of the function between the ranges of the variable is the only method for the purpose.

Take, for example, the continuous function $y(x)$ in Figure 2.43a; the average value of this function over the range $x = a$ and $x = b$ can be determined by

$$y_{av} = \frac{\text{Area of rectangle } (A)}{\text{Base of the rectangle}}$$

in which area (A) is the area bounded by the function $y(x)$ in Figure 2.43a between $x = a$ and $x = b$. The same area is equal to the area of the rectangle in Figure 2.43b with the base $(b - a)$. We may thus express the above relationship in the form

$$y_{av} = \frac{\int_a^b y(x)\,dx}{b - a} \tag{2.23}$$

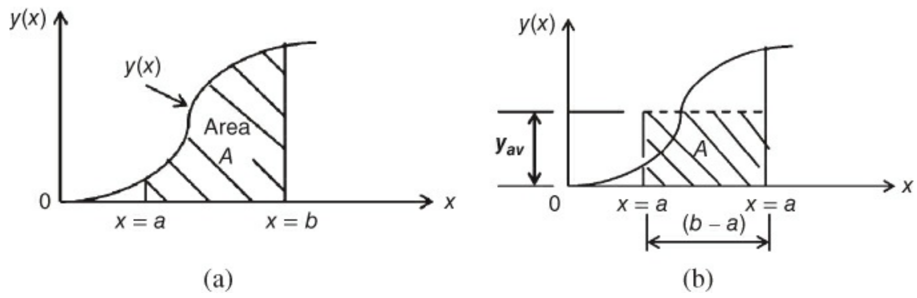

Figure 2.43 Average value of a continuous function. (a) A continuous function. (b) The average value of the function.

Example 2.13

A meteorologist recorded ambient temperature at the San Jose International Airport at the following three time points during a day:

Time of day	Ambient temperature readings (°F)
10:00 a.m.	72
2:00 p.m.	82
6:00 p.m.	65

Estimate the following:

a. The average temperature of the day between 6:00 a.m. and 9:00 p.m.

b. The maximum temperature during that period.

Solution:

We realize that the change of ambient temperature at a specific location is a continuous phenomenon. As such, we need to derive a continuous function that will include the three measured values for the required answers.

We may choose to express the three measured ambient temperatures on a 24-hour clock basis as follows:

Hour of the day (based on 24-hour/day clock)	Measured ambient temperature (°F)
10	72
14	82
18	65

We will use a polynomial function to fit the curve that will include the three measured temperatures and will have the form

$$T(t) = a_0 + a_1 t + a_2 t^2 \tag{a}$$

where $T(t)$ is the continuous function representing the ambient temperature at variable time t, and a_0, a_1, and a_2 are coefficients to be determined by the measured values of temperature in the above table.

Following the procedures presented in Section 2.2.4, we may determine the values of the three coefficients to be

$$a_0 = -71.1, \qquad a_1 = 22.75, \qquad a_2 = -0.844$$

which leads to the function $T(t)$ in the form

$$T(t) = -71.1 + 22.75t - 0.844t^2 \tag{b}$$

The function in Equation (b) allows us to either interpolate or extrapolate temperatures at the specific location that are respectively within or outside the range of the measured temperature.

a. The average temperature of the day between 6:00 a.m. and 9:00 p.m. (or the 21st hour of the day) can thus be estimated using Equation (2.23):

$$T_{av} = \frac{\int_6^{21} T(t)\,dt}{21-6} = \frac{\int_6^{21} (-71.1 + 22.75t - 0.844t^2)\,dt}{15} = 66.38°\text{F}$$

b. The time and the value of the maximum temperature of the day may be computed in the following way:

Since we have already derived a continuous function for the temperature variation of the day as shown in Equation (b), we may use on Equation (b) the formula from calculus for determining the maximum value of a continuous function $f(x)$.

Thus, we solve for the value of the associate variable by letting x_m be the value at which the function $f(x)$ has a maximum or minimum value by letting $df(x)dx = 0$, and finding a maximum value at x_m if

$$\left.\frac{d^2f(x)}{dx^2}\right|_{x=x_m} < 0$$

By following the first step, we may determine that a maximum or minimum ambient temperature of 82.2°F occurs at time t = 13.48 hour of the day. Evaluation of the second derivative of the temperature function in Equation (b) with t = 13.48 shows a value that is less than zero, which indicates that the ambient temperature of 82.2°F indeed is the maximum ambient temperature of the day at the location, and it occurs at 13.48 hour or 1:48 p.m.

2.4 Special Functions for Mathematical Modeling

Engineering students at the junior or senior levels of their undergraduate studies in many engineering schools will already have gained good experience in handling standard functions such as trigonometric and exponential functions in their mathematical education. However, there are engineering analyses that involve situations requiring mathematical formulas to describe physical phenomena that can only be represented by special mathematical functions. These are referred as "special functions" in engineering analysis because they do not appear in many mathematical analyses as commonly as the trigonometric and exponential functions.

There are generally two types of special functions used in engineering analysis:

- **Type 1. Special functions appear in mathematical formulations and solutions.** We will introduce three common special functions that often appear in solutions of engineering analysis:

    error function and complementary error functions;

    gamma function;

    Bessel functions.

- **Type 2. Special functions describing special physical phenomena.** These functions are useful tools in mathematical modeling of two particular physical phenomena that often present in engineering analyses. We will introduce two such special functions:

    step functions;

    impulsive functions.

2.4.1 Special Functions in Solutions in Mathematical Modeling

2.4.1.1 The Error Function and Complementary Error Function

This function often appears in the solution of differential equations such as diffusion equations. Diffusion is the net movement of a substance (e.g., an atom, ion, or molecule) from a region of high concentration to a region of low concentration and is a common phenomenon in nature. Oxidation of metals in a moist environment is a typical example. We will look at a special case of diffusion analysis in which the variation of the concentration of a solvent, solvent A, with concentration C_1 diffuses into another solvent, solvent B, with a lower concentration C_2, as illustrated in Figure 2.44.

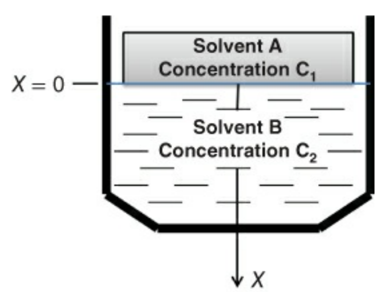

Figure 2.44 Diffusion of substance A into solvent B.

The concentration of solvent A in the mixed solvent at the depth x and time t after its diffusion into solvent B, expressed as $C(x,t)$, can be obtained using the differential equation already presented in Equation 2.5 (Hsu, 2008):

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2} \tag{2.5}$$

in which D is the diffusivity of solvent A into solvent B. Diffusivity is a material constant for many substances involved in diffusion processes and the values are available from materials handbooks. The following conditions are specified for the present case of analysis:

$$C(x,0)=0;\quad C(0,t)=C_s;\quad C(\infty,t)=0$$

where C_S is the concentration of solvent A at the interface of the two solvents and $C(\infty,t)$ is the concentration of solvent in the mixed solvent far from the interface at time t.

The solution of Equation 2.5 with the specified conditions is

$$C(x,t)=C_s\,\mathrm{erfc}\left(\frac{x}{2\sqrt{Dt}}\right) \tag{2.24}$$

The function $\mathrm{erfc}(x/2\sqrt{Dt})$ or $\mathrm{erfc}(X)$ with $X=x/2\sqrt{Dt}$ in Equation 2.24 is called the complementary error function, which is related to the error function $\mathrm{erf}(X)$ by the relationship $\mathrm{erfc}(X) = 1 - \mathrm{erf}(X)$. The error function is given by

$$\mathrm{erf}(X)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt \tag{2.25}$$

Just as for the sine and cosine functions, the value of the error function $\mathrm{erf}(x)$ can be determined from its argument X. Values of the error function with given argument X are available in many textbooks and handbooks (Hsu, 2008; Zwillinger, 2003). It is, however, convenient to recognize that $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(\infty) = 1$.

2.4.1.2 The Gamma Function

Like the error function, the gamma function $\Gamma(t)$ also appears in solutions of mathematical modeling. It takes the form

$$\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}\,dx \tag{2.26}$$

The gamma function with integer argument t in Equation (2.26) can be evaluated by the simple formulas

$$\Gamma(t+1)=t\Gamma(t)\quad\text{and}\quad\Gamma(n)=(n-1)!$$

The sign "!" designates the "factorial" value of an integer number. Values of gamma functions are available in mathematics handbooks such as Zwillinger (2003).

2.4.1.3 Bessel Functions

Bessel functions are special functions that often appear in engineering analyses involving problems of "circular," "cylindrical," and "spherical" geometry. They are named after Friedrich Bessel, a German astronomer and mathematician (1784–1846). Bessel developed the Bessel functions in 1824 from his study of the elliptical motion of planets.

Bessel functions behave in a similar way to the periodic sine and cosine functions except that they oscillate with gradual reduction of both the amplitude and frequency.

Bessel Equation and Bessel Functions

Bessel functions are solutions of the Bessel equation, which has the form

$$x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}+(\lambda^2x^2-n^2)y(x)=0 \tag{2.27}$$

where λ = constant and n = the integers 1, 2, 3,….

The solution of the Bessel equation (2.27) has the form

$$y(x) = C_1 J_n(\lambda x) + C_2 Y_n(\lambda x) \quad \mathbf{2.28}$$

in which C_1 and C_2 are arbitrary constants.

The functions $J_n(\lambda x)$ and $Y_n(\lambda x)$ are Bessel functions with specific names:

$J_n(\lambda x)$ is the Bessel function of the first kind of order n.

$Y_n(\lambda x)$is the Bessel function of the second kind of order n; it is often called a Neumann function.

Figures 2.45a and b illustrate plots of the first two orders of Bessel functions. We will notice that the values of Bessel functions oscillate about the x-axis but with gradual reduction of amplitudes and periods of oscillation.

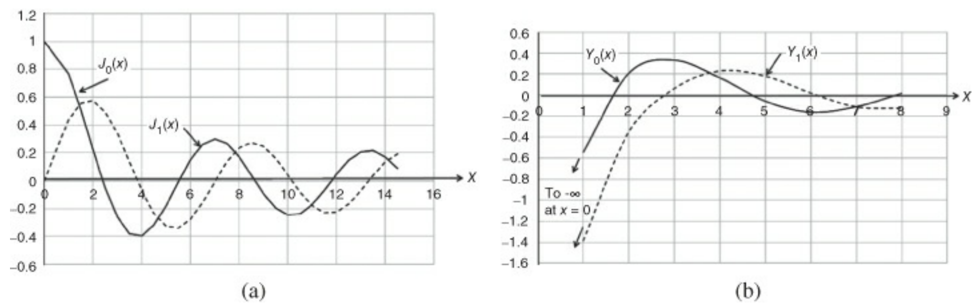

Figure 2.45 Bessel functions. (a) Bessel functions $J_0(x)$ and $J_1(x)$. (b) The Neumann functions $Y_0(x)$ and $Y_1(x)$.

A slightly different form of the Bessel equation from Equation 2.27 is

$$x^2 \frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx} - (\lambda^2 x^2 + n^2)y(x) = 0 \quad \mathbf{2.29}$$

The solution of the modified Bessel equation in Equation 2.29 is

$$y(x) = C_1 I_n(\lambda x) + C_2 K_n(\lambda x) \quad \mathbf{2.30}$$

in which $I_n(\lambda x)$ is a modified Bessel function of the first kind of order n, and $K_n(\lambda x)$ is a modified Bessel function of the second kind of order n.

It may be noted from Figure 2.46 that the modified Bessel functions in Equation 2.30 do not oscillate as do the Bessel functions in Equation 2.28.

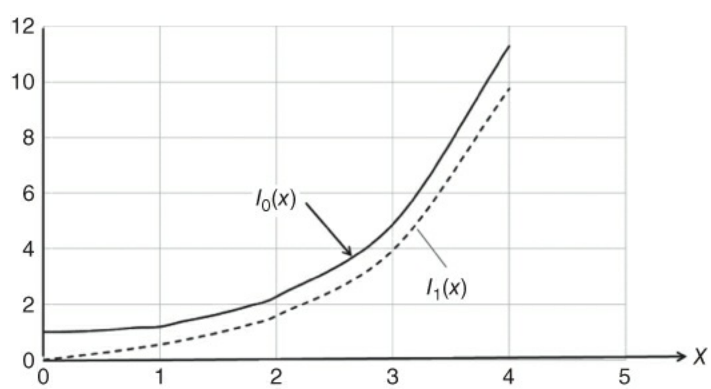

Figure 2.46 Graphical illustration of modified Bessel functions $I_0(x)$ and $I_1(x)$.

Example 2.14

Solve the following differential equation:

$$x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}+(9x^2-4)y(x)=0$$

We recognize that this differential equation is similar in form to Equation 2.27 with $\lambda^2 = 3^2$, and $n^2 = 2^2$. We may thus express the solution as that shown in Equation 2.28 with

$$y(x)=C_1J_2(3x)+C_2Y_2(3x)$$

The constants C_1 and C_2 in the above solution may be determined according to specified conditions.

Likewise, the solution of the modified Bessel equation

$$x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}-(9x^2+4)y(x)=0$$

may be expressed in the form of Equation 2.30:

$$Y(x)=C_1I_2(3x)+C_2K_2(3x)$$

Values of Bessel functions of order 0 and 1 with specified arguments are available in mathematical handbooks such as Zwillinger (2003), and as illustrated in Figure 2.45. For instance, $J_0(10.5) = -0.23664$, $J_1(10.5) = -0.07885$, $J_2(10.5) = 0.22162$, and $Y_0(11) = -0.16884$, $Y_1(11) = 0.16370$, $Y_2(11) = 0.19861$. It is useful to be able to recognize the values of Bessel functions of certain special orders and arguments, for instance:

$$J_0(0)=1,\quad J_n(0)=0,\quad Y_n(0)\to\infty$$

$$I_0(0)=1,\quad I_n(0)=0,\quad K_n(0)\to\infty$$

Recurrence Relations of Bessel Functions

One will find that many handbooks offer only the values of Bessel functions of order 0 and 1; Bessel functions of higher orders can be obtained by using the following recurrence relations:

$$J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x) \tag{2.31}$$

$$Y_{n-1}(x)+Y_{n+1}(x)=\frac{2n}{x}Y_n(x) \tag{2.32}$$

Thus, for example, the value of function $J_2(x)$ can be determine by letting $n = 1$ in Equation 2.31 to get

$$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$$

with the values of $J_0(x)$ and $J_1(x)$ being available in mathematical handbooks (Zwillinger, 2003).

The modified Bessel functions $I_n(x)$ and $K_n(x)$ may be related to the Bessel functions by the formulas:

$$I_n(ix)=i^nJ_n(x)\quad\text{and}\quad I_n(x)=i^{-n}J_n(x)$$

where $i=\sqrt{-1}$.

$$K_n(ix) = \frac{\pi i}{2} e^{-(n\pi/2)i}[-J_n(x) + iY_n(x)]$$

Differentiation and Integration of Bessel Functions

The following expressions are used for differentiation:

$$\frac{d}{dx}J_n(ax) = \frac{n}{x}J_n(ax) - aJ_{n+1}(ax) \qquad \mathbf{2.33}$$

$$\frac{d}{dx}Y_n(ax) = \frac{n}{x}Y_n(ax) - aY_{n+1}(ax) \qquad \mathbf{2.34}$$

and for integration:

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) \qquad \mathbf{2.35}$$

2.4.2 Special Functions for Particular Physical Phenomena

2.4.2.1 Step Functions

Step functions originated from "Heaviside step functions." They are used to describe physical phenomena that come to existence at a given instant in a "time" domain, or at a position in the "space" domain. These physical phenomena remain in their original state thereafter. Physical phenomena of this kind may be a "weight" or "force" existing in a portion of a solid structure, or for the switching on or off of an electric circuit, resulting in the electric current beginning to flow or ceasing to flow in the circuit thereafter.

The physical situation that a step function represents can be represented diagrammatically as in Figure 2.47. The mathematical expression of the step function in Figure 2.47 is

$$\begin{aligned} u(x-a) \ \text{or} \ u_a(x) &= 0 \quad \text{for} -\infty < x < a \\ &= \alpha \quad \text{for} \ a < x < \infty \end{aligned} \qquad \mathbf{2.36a}$$

and

$$u(x-a) = u_a(x) \to \frac{\alpha}{2} \quad \text{for} \ x = a \qquad \mathbf{2.36b}$$

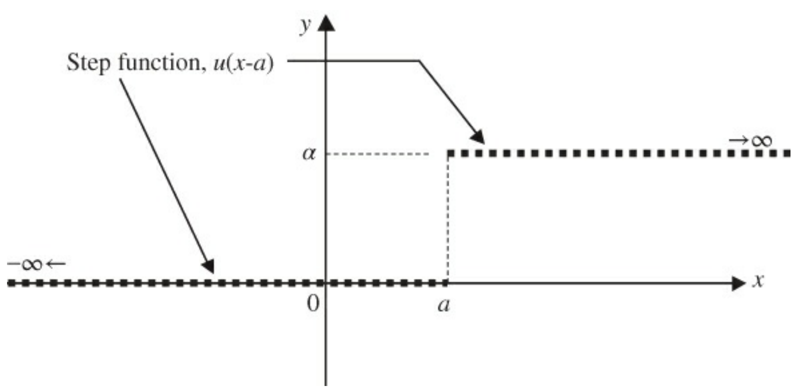

Figure 2.47 Graphical description of a step function.

The function $u_a(x)$ in Equations 2.36a,b denotes a step function with non-zero values, which begins at $x = a$. Step functions can be added or subtracted to represent physical quantities existing over infinite range of variables from $-\infty < x < \infty$ as in Equations 2.36a,b

Example 2.15

Use a step function to describe a physical quantity that varies in the form of "rectangular wave" as illustrated in Figure 2.48.

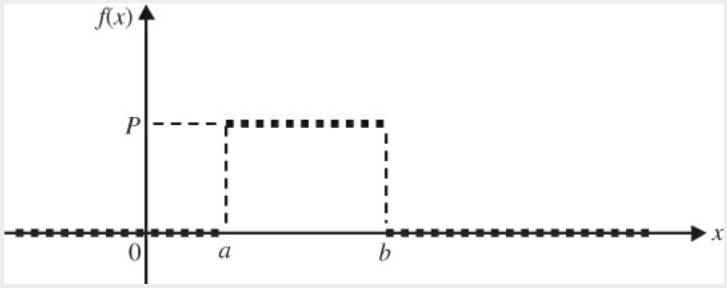

Figure 2.48 General form of a function existing in a finite range in an infinite variable domain.

Solution:

The situation in Figure 2.48 can be described by the superposition of two step functions as illustrated in Figure 2.49. Thus, a physical quantity existing between $x = a$ and $x = b$ in an infinite variable domain such as shown Figure 2.48 can be modeled mathematically by the following expression:

$$\begin{aligned} f(x) &= f_1(x) - f_2(x) \\ &= Pu(x-a) - Pu(x-b) \\ &= P[u(x-a) - u(x-b)] \\ &= Pu_a(x) - Pu_b(x) \end{aligned} \tag{2.37}$$

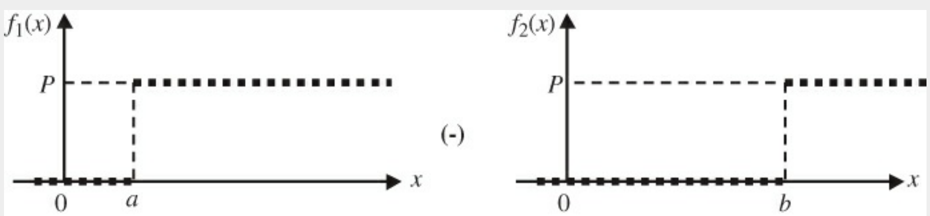

Figure 2.49 Superposition of two step functions.

Example 2.16

Use a step function to describe a beam that is subjected to a uniformly distributed load of intensity $w(x)$ over part of its length, as illustrated in Figure 2.50 so that the function describing the load on the beam is valid for the range $-\infty < x < +\infty$.

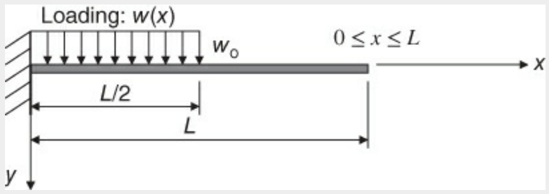

Figure 2.50 A cantilever beam subjected to partial distributed loading.

Solution:

The beam is subjected to a uniformly distributed load $w(x)$ with an intensity w_0 , as shown in Figure 2.50, whereas the other part of the beam is load free. The loading function is in the shape of a "step" with the height of the step being w_0 and on the span defined by $0 \leq x \leq L/2$. The loading for the *entire* beam can be described by a step function that is equal to the difference of two step functions with same "height" of the steps, i.e., w_0, but with one function beginning at $x = 0$ and the other with the "step" beginning at $x = L/2$, as illustrated in Figure 2.51.

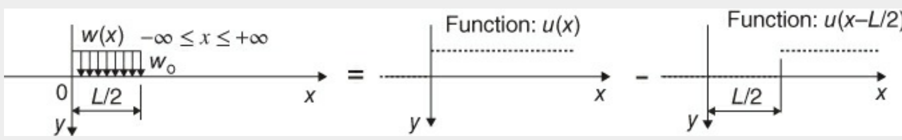

Figure 2.51 Application of a step function to a partially loaded beam.

The loading function for the entire beam length is thus

$$w(x) = w_0 \left[u(x) - u\left(x - \frac{L}{2}\right) \right] \quad \text{with} \quad -\infty \leq x \leq +\infty \tag{2.38}$$

2.4.2.2 Impulsive Functions

Unlike the step function that describes a physical quantity that comes to existence at a specific time or spatial location and remains at its original value thereafter, the *impulsive function* is used to describe physical phenomena that exist only for an *extremely short period of time* or in an extremely small physical extent (i.e., localized physical phenomena). Impulsive functions have other names such as the "delta function" or the "Dirac function." A graphic representation of this function is illustrated in Figure 2.52.

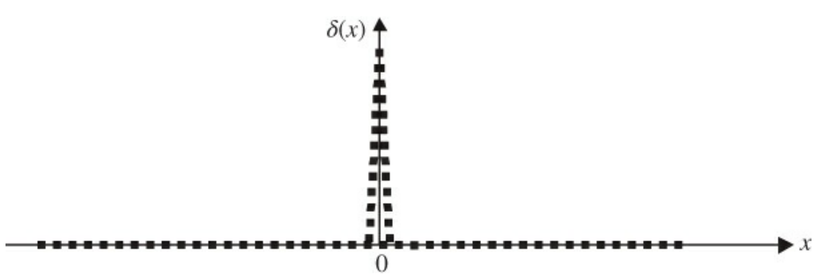

Figure 2.52 Graphical definition of impulsive function.

The mathematical expression of the impulsive function is

$$\delta(x) = 0 \quad \text{for } x \neq 0$$
$$\rightarrow \infty \quad \text{for } x = 0 \qquad \textbf{2.39}$$

In practice, there is no physical quantity that has an infinite magnitude along with zero pulse width, as illustrated in Figure 2.52. Therefore, for real physical phenomena we deal with finite magnitude and finite but very small pulse width, as illustrated in Figure 2.53. The corresponding function must be modified to be for the following cases.

a. The pulse occurs at the origin (Figure 2.53):

$$\begin{aligned} \delta(x) &= 0 \quad x \leq 0 \\ &= 1/\varepsilon \quad 0 < x \leq \varepsilon \\ &= 0 \quad x > 0 \end{aligned} \qquad \textbf{2.40}$$

b. The pulse occurs at a point $x = a$ away from the origin (Figure 2.54):

$$\begin{aligned} \delta(x-a) \text{ or } \delta_a(x) &= 0 \quad x \leq a \\ &= 1/(\varepsilon - a) \quad a < x < \varepsilon \\ &= 0 \quad x \geq \varepsilon \end{aligned} \qquad \textbf{2.41}$$

where $\delta_a(x)$ is an impulsive function with the pulse located at $x = a$.

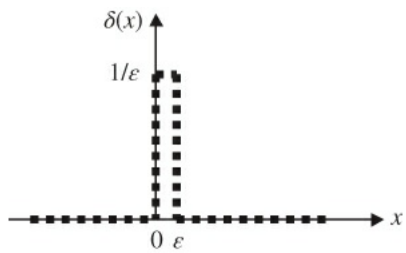

Figure 2.53 Impulse at origin of a coordinate system.

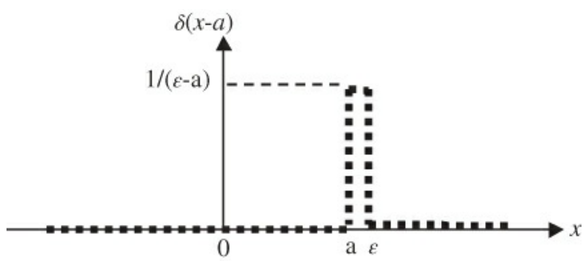

Figure 2.54 Off-origin impulsive function.

The following are some useful properties of impulsive functions in mathematical modeling:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad \textbf{2.42a}$$

$$\int_{0}^{t} \delta(t-a)\,dt = u(t-a) \qquad \textbf{2.42b}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a) \qquad \textbf{2.42c}$$

$$\frac{du(t)}{dt} = \delta(t) \qquad \textbf{2.42d}$$

Example 2.17

Use an impulsive function to describe the concentrated loading P_0 on a cantilever beam as shown in Figure 2.55.

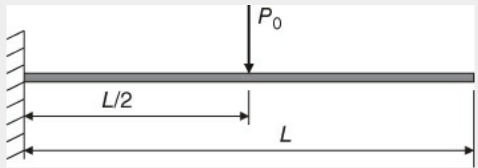

Figure 2.55 Beam subjected to a concentrated load.

Solution:

Because the load is a concentrated force that applies at a "spot" located at $x = L/2$, we may use an impulsive function to describe this type of loading as illustrated in Figure 2.56 with a pulse of height P_0 covering the range of $-\infty \leq x \leq +\infty$.

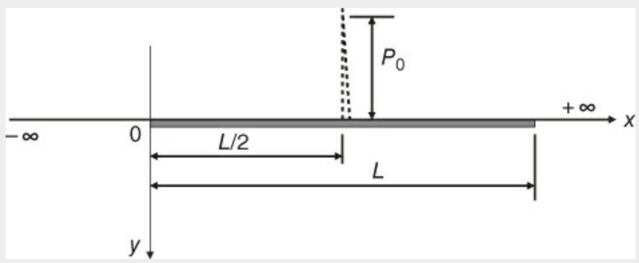

Figure 2.56 Impulsive function with a pulse P_0.

The loading function that covers the variable domain (x) defined by $-\infty \leq x \leq \infty$ may be expressed by an impulsive function as

$$P(x) = P_0\delta(x - L/2) \tag{2.43}$$

2.5 Differential Equations

We began this book with the statement that “Engineering analysis involves the application of scientific principles and approaches … to reveal the physical state of an engineering system, a machine or device, or structure under study.” This properly reminds engineers that the laws of physics should be used as the basis of all engineering analyses. Consequently, all mathematical modeling in engineering analysis must adhere to the laws of physics.

2.5.1 The Laws of Physics for Derivation of Differential Equations

There are three universal laws of physics that all principles and approaches of engineering analysis must follow:

1. Conservation of mass
2. Conservation of energy
3. Conservation of momentum.

Many “application” laws of physics have been derived from these three fundamental laws of physics. For mechanical engineering analyses, the following application laws are often used as bases for the derivation of differential equations used in engineering analysis such as:

- Newton's laws for the mechanics of solids in static, dynamic, and kinematic analyses of machines and rigid bodies.
- Fourier's law for the conduction of heat in solids (it relates the “heat flows” and the induced “temperature gradients,” or vice versa, in solids).
- Newton's cooling law for convective heat transfer in fluids.
- Bernoulli's law for fluid dynamics.

The following examples illustrate how other laws of physics can be used to derive differential equations in mathematical modeling.

Example 2.18

Derive an equation to determine the variation of weight of a freely hung solid cone along its length. The cone has root radius R and length L as illustrated in Figure 2.57.

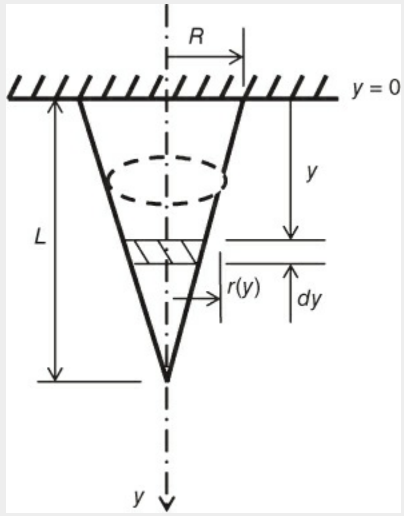

Figure 2.57 Freely hung solid cone.

Solution:

We realize that the weight, W, of a solid can be determined by the product of the volume of the solid (V) and its mass density (ρ): or $W = \rho V$. A solution to the gradual weight increase of the cone along its length can be determined by the variation of its volume along its length. Our task is thus to deriving an equation to determine the volumetric variation of the cone along its length.

Referring to Figure 2.57, the volume of the cone can be obtained by summing up all the element volume in the cross-hatched area in the figure. At an arbitrary distance from the fixed end at $y = 0$, the volume of the shaded element equals $\Delta V = \pi[r(y)]^2 \Delta y$, in which Δy is the thickness of the volume element.

We may relate the radius $r(y)$ to the variable y by

$$r(y) = R\left(1 - \frac{y}{L}\right)$$

from which we may formulate the volume of the element:

$$\frac{\Delta V}{\Delta y} = \pi R^2 \left(1 - \frac{y}{L}\right)^2$$

We also realize that the volume of the cross-hatched element increases continuously with the increment of y in the last expression. Mathematically, this validates the condition that the relation holds under the condition that Δy is infinitesimally small, or $\Delta y \rightarrow 0$, which leads to the following differential equation:

$$\frac{dV(y)}{dy} - \pi R^2 \left(1 - \frac{y}{L}\right)^2 = 0 \qquad \textbf{2.44}$$

with the condition $V(0) = 0$.

Example 2.19

Derive the equation of motion of a falling ball of mass *m* as illustrated in Figure 2.58.

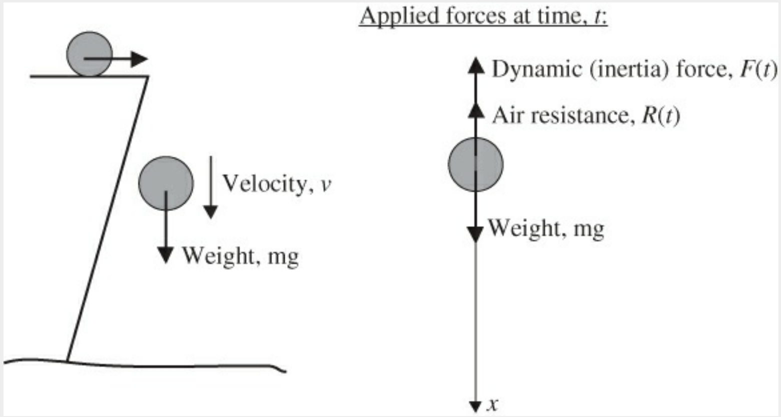

Figure 2.58 Forces on a free fall rigid body.

Solution:

We realize that the ball falls from zero initial velocity and falls with a constant gravitational acceleration g toward the ground. Due to this gravitational acceleration, the velocity of the falling ball v increases with time t.

If we let $v(t)$ be the velocity of the mass at time t, the forces that act on the ball at any moment t are as shown in the right-hand part of Figure 2.58. The dynamic equilibrium condition requires that the sum of all these forces be set to zero according to Newton's first law. Mathematically, this is expressed as

$$\sum F_x = -F(t) - R(t) + mg = 0 \tag{a}$$

where $F(t)$ = dynamic (inertial) force, which is in the opposite direction to the acceleration (g in this case) according to Newton's second law; $R(t)$ = the force of air resistance from the air the ball is passing through is proportional to the velocity, or $R(t) = cv(t)$ where c is a proportionality constant; mg = the weight of the ball.

The dynamic force acting on the falling ball can be expressed by Newton's second law as $F(t) = M\, a(t)$, where $a(t)$ = the gravitational acceleration (in this case), which can be expressed as

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = \frac{d^2x(t)}{dt^2}$$

where $x(t)$ is the instantaneous position of the falling ball.

By substituting the expressions for all the forces into Equation (a), we may derive the following differential equation to be solved for the ball's instantaneous position, $x(t)$, and from which we may obtain the corresponding instantaneous velocity $v(t)$.

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} - mg = 0 \tag{2.45}$$

Problems

2.1 What is the difference between a real number and an imaginary number, and between a constant and a parameter?

2.2 What is mathematical modeling and why does it play a vital role in engineering analysis?

2.3 How would you select an appropriate function with appropriate associated variables that could be used in the solution of the following physical problems?

a. The time required to drain a swimming pool 20 m × 40 m × 2 m deep.

b. The time required to transport a silicon wafer by a moving chuck from one end to the other end of a platform of given dimensions.

c. The time required for a large rotor of an electricity generator made of steel to be heated up to 500°C from room temperature in a furnace in a heat treatment process.

d. The position of your car determined by a GPS (global positioning system).

e. The amplitude of vibration of a mass attached to a spring after the application of a small but instantaneous disturbance to the initially motionless mass.

f. The bending stress in a cantilever beam subjected to a concentrated load (also formulate the function and variable representing the bending stress in a diving board over a swimming pool).

2.4 Give at least one "real-world" situation that can be represented by (a) function of discrete values, and (b) continuous functions.

2.5 Why do we need to use Equation 2.23 to determine the average value of a continuous function?

2.6 Use no more than 25 words to outline the application of curve fitting techniques in engineering analysis.

2.7 Use no more than 25 words to describe the application of the step function and the impulsive function in engineering analysis.

2.8 What are the principal applications of Bessel functions in mathematical modeling of engineering problems?

2.9 Describe geometrically the families of lines represented by the following functions: (a) $y = mx-3$, and (b) $y = 4x + b$, where m and b are real numbers.

2.10 If $f(x) = x^2-x$, show that $f(x+1) = f(-x)$.

2.11 Draw the graph of the function

$f(x) = x-1$ if $0 < x < 1$

$f(x) = 2x$ if $x \geq 1$.

2.12 Find the derivatives of each of the following functions:

a. $y = 1/x^2$

b. $y = (1 + 2x)^{1/2}$

c. $y = 1/(2 + x)^{1/2}$

2.13 Derive an expression for the temperature $T(r)$ in a nuclear reactor vessel wall as illustrated in Figure 2.59. The temperature across the reactor wall fits an exponential function with its inner wall maintained at 400°C and the outer wall at 100°C. Constants a and b in Figure 2.59 are the respective inner and outer radii of the wall of the reactor vessel.

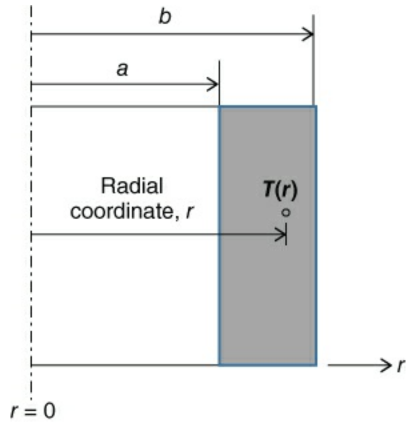

Figure 2.59 Temperature in a nuclear reactor vessel wall.

2.14 Find the average ordinate of (a) semicircle of radius r; (b) the parabola = $4 - x^2$ from $x = -2$ to $+2$.

2.15 Temperature is measured in either Fahrenheit or Celsius degrees. Fahrenheit (F) and Celsius (C) temperature are related by the expression $F = aC + b$ in which a and b are constants. The freezing point of water is 0°C or 32°F, and the boiling point of water is 100°C or 212°F. (a) Find the equation relating C and F. (b) At what temperature will the temperatures on both scales be equal in value?

2.16 Derive a function for the variation of the radius along the length of a tapered metal rod with dimensions shown in Figure 2.60 below:

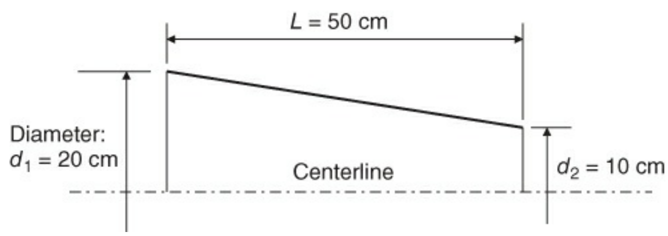

Figure 2.60 Variation of diameter of a rod.

2.17 Find the variation of the cross-sectional area, the volume, and the mass of the tapered rod in Figure 2.60 along its length if the mass density of the rod material is ρ g/cm^3.

2.18 A common shape of pressure vessel used in industry involves a hollow cylinder covered at both its ends with welded "2:1 elliptical covers" (or "heads") as illustrated in Figure 2.61. (a 2:1 elliptical head means that the major axis of the ellipse is twice that of the minor axis). Use the integration method to determine the volume content of the pressure vessel with the interior dimensions shown in the figure.

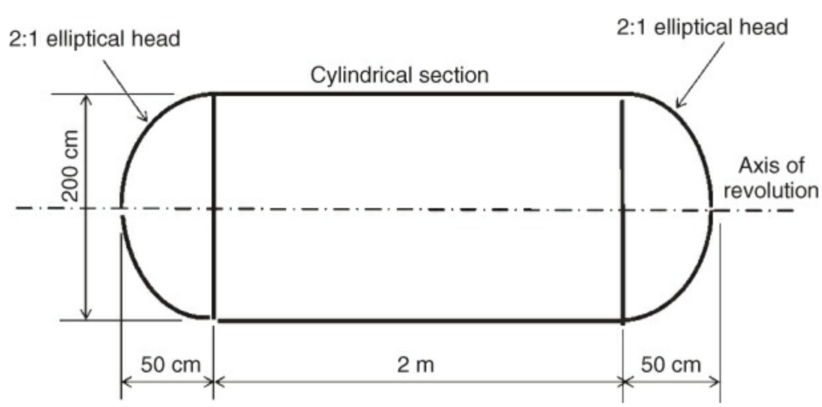

Figure 2.61 A Pressure vessel containing liquid.

2.19 An engineer designs a circular filling funnel for a water bottling company with a filling volume of 8 fluid ounces (or 236 cm^3). The filling funnel consists of three sections as illustrated in Figure 2.62. Determine the approximate tapering angle θ and the lengths L_1 and L_2 that satisfy a design constraint on available space that requires $L_1 + L_2 \leq 10$ cm.

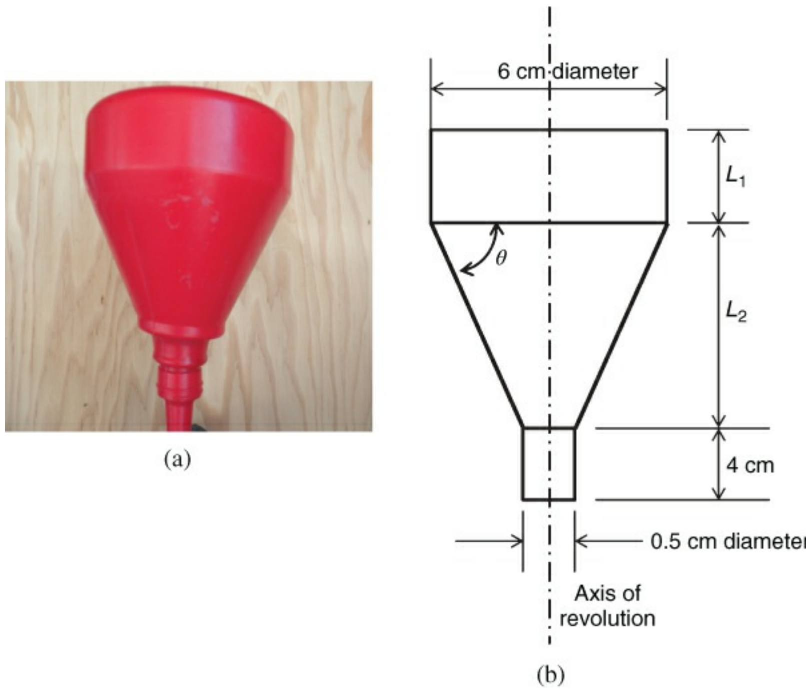

Figure 2.62 Optimal design of a three-section funnel. (a) Image of the three-section funnel. (b) Dimensions of the funnel.

2.20 Use an integration method to determine the volume content of a square tapered chute as illustrated in Figure 2.63 with dimensions W = 20 cm, H = 10.8 cm, and h = 1.2 cm.

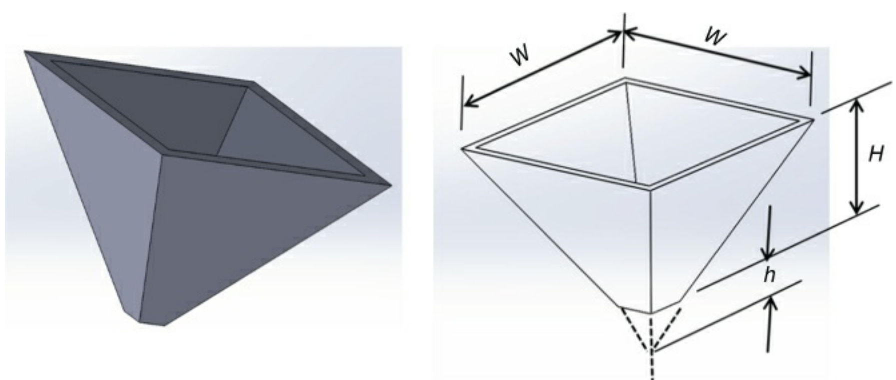

Figure 2.63 Square tapered chute.

2.20 (Hint: You may check your answer with the formula for the volume of a pyramid from handbooks: Volume of 4-side base pyramids V = (base area)(altitude)/3.)

2.21 A measuring cup shown in Figure 2.64a has overall dimensions given in Figure 2.64b. Determine the following: (a) the overall volume of the cup; (b) the height L_1 for the volume of 200 ml; and (c) the height L_2 for the volume of 100 ml.

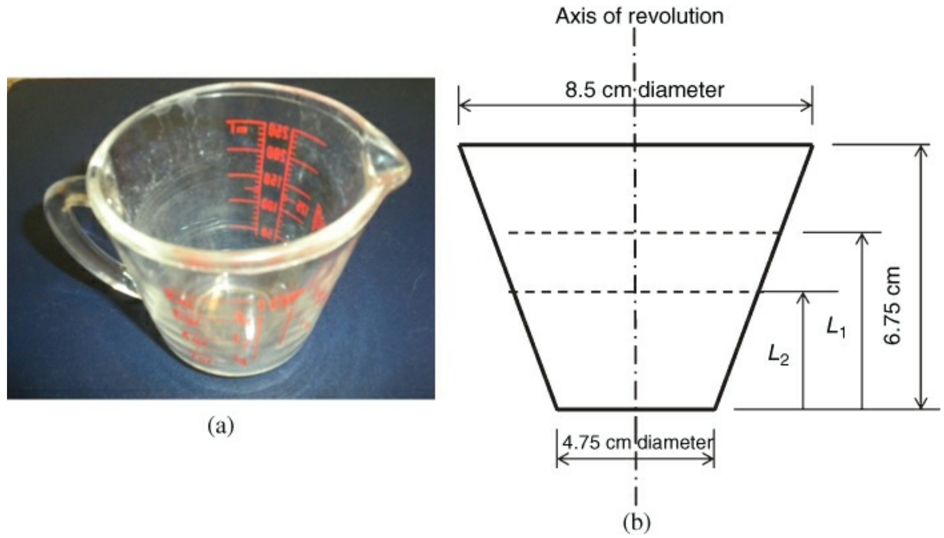

Figure 2.64 Design of a measuring jug. (a) Image of the measuring jug. (b) The overall dimensions of the jug.

2.22 IV bottles such as the one shown in Figure 2.65 are commonly used in hospitals. Use the idealized geometry of the IV bottle shown at the right of the figure to determine

a. The volume of the liquid in the bottle.

b. The diameter of the straight portion of the bottle in Figure 2.65 below if the capacity is designed to be 1200 ml (cm^3).

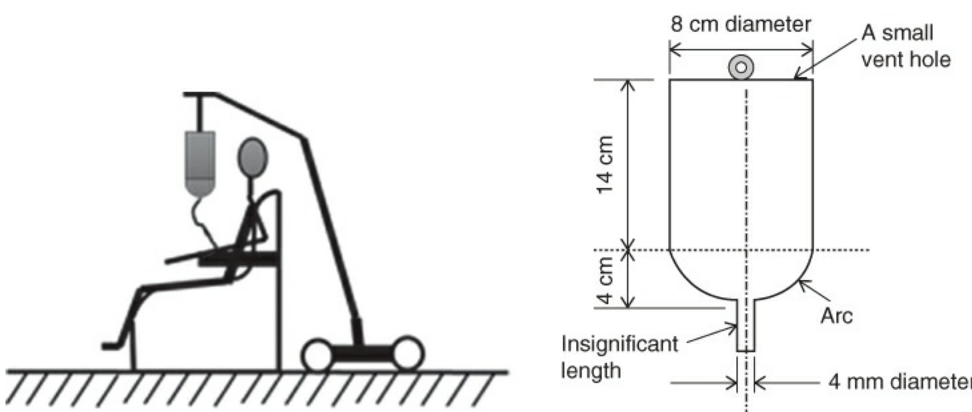

Figure 2.65 IV bottle for hospital use.

2.23 Use appropriate step functions to describe the loading on the beam illustrated in Figure 2.66.

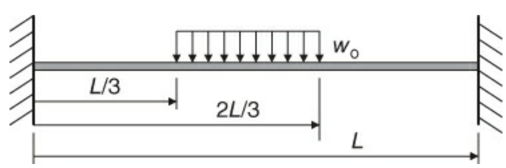

Figure 2.66 A partially loaded beam.

2.24 Use appropriate step functions to describe the loading on the beam illustrated in Figure 2.67.

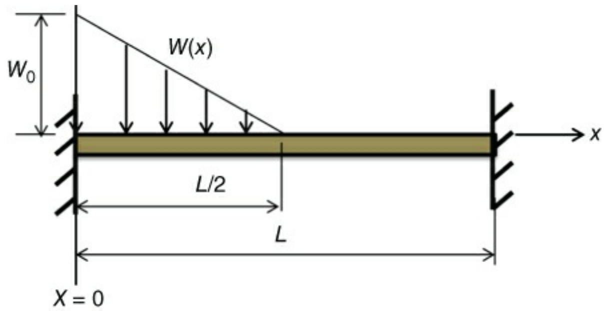

Figure 2.67 Nonuniformly loaded beam.

2.25 Use the step function to express the loading function $P(x)$ on a partially loaded beam so that the loading function is valid for the region $-\infty < x < +\infty$. The loading condition on the beam is illustrated in Figure 2.68.

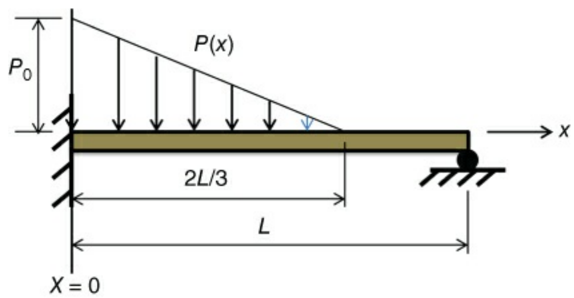

Figure 2.68 Partially loaded beam.

2.26 Use an appropriate special function to describe the loading on the beam illustrated in Figure 2.69.

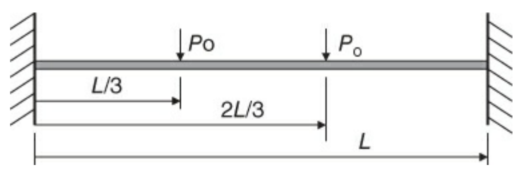

Figure 2.69 A beam subjected to concentrated loads.

2.27 Derive a mathematical expression for the distributed loads, $P(x)$ on the beam illustrated in Figure 2.70.

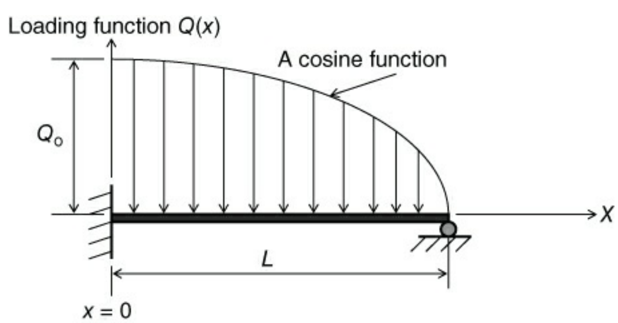

Figure 2.70 A cantilever beam subjected to a distributed load.

2.28 Use the integration method to locate the centroid of a flat plate bounded by three corners A, B, and C with the dimensions shown in Figure 2.71.

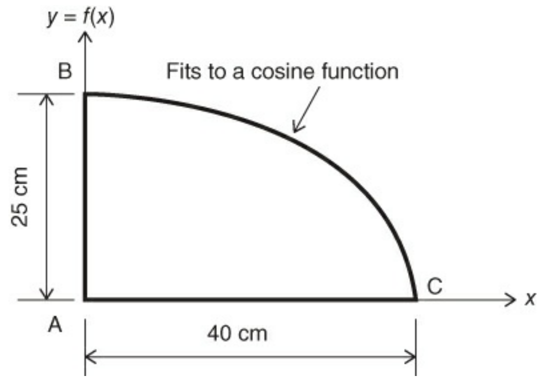

Figure 2.71 A plate with a curved edge.

2.29 Use the integration method to locate the centroid of a plate made of a quarter-circle as illustrated in Figure 2.72.

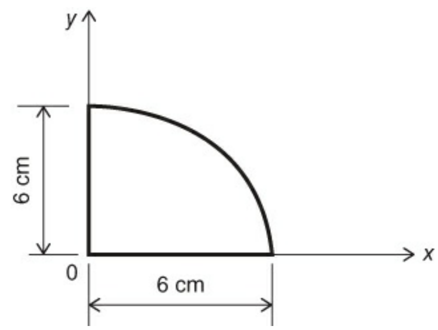

Figure 2.72 A plate of quarter-circle geometry.

2.30 Use the integration method to locate the centroid of a quarter-ellipse as illustrated in Figure 2.73.

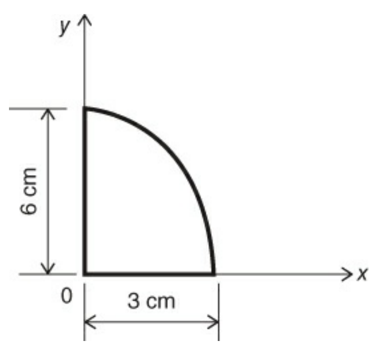

Figure 2.73 A plate of quarter-ellipse geometry.

2.31 Use the integration method to locate the centroid of a plate of the geometry and the dimensions shown in Figure 2.74.

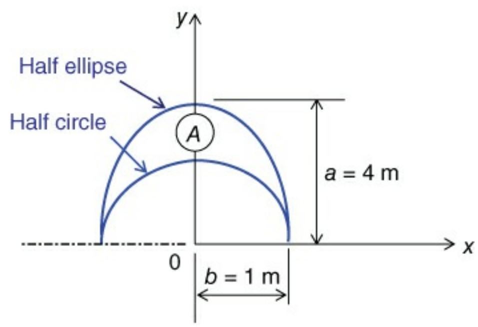

Figure 2.74 A plate of geometry bounded by a half ellipse and a half circle.

2.31 *Hint:* The use of "double integration" technique in determining the enclosed area and area moments of the plane area along both x- and y-coordinates will prove to be a proficient way to reach solutions of this problem

2.32 Use the integration method to determine the area and the location of the centroid of a plate coupler of a four-bar linkage similar to that illustrated in Figure 2.36. This coupler plate has dimensions defined by the coordinates of the four corners ABCD as shown in Figure 2.75.

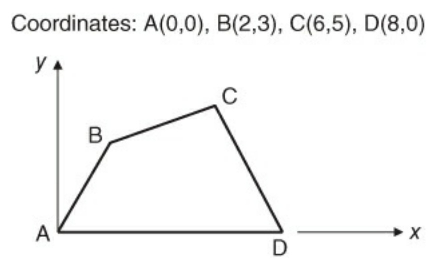

Figure 2.75 Plate coupler of a four-bar linkage.

2.33 According to a textbook on dynamics (Hibbeler, 2007), the (mass) moment of inertia (I) of a solid machine component is an important quantity that is used as a measure of the resistance of a rigid body to angular acceleration ($\boldsymbol{\alpha}$) caused by moment $\mathbf{M} = I\boldsymbol{\alpha}$, in which the boldface characters indicate vectorial quantities. The mass moment of inertia of a rigid body can be determined by an integral:

$$I = \int_v r^2 \rho \, dv \qquad \textbf{a}$$

where r is the perpendicular distance of the moment arm from the axis of rotation, ρ is the mass density of the rigid body, and v is the volume of the rigid body.

2.33 Use the above expression to determine the mass moment of inertia of a flywheel with a linearly tapered profile. The flywheel is made of aluminum with a mass density of 2.7 g/cm^3. The cross-section of the wheel is shown in Figure 2.76. Determine the mass moment of inertia of this flywheel using the integral (Equation a) shown above.

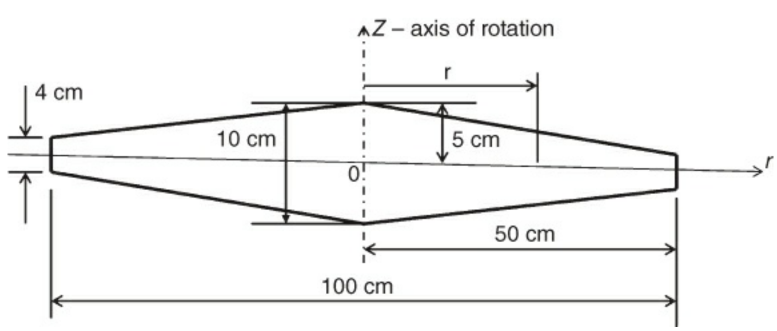

Figure 2.76 Cross-section of a flywheel with a tapered profile.