Chapter IX
DIFFERENTIAL EQUATIONS

Sec. 1. Verifying Solutions. Forming Differential Equations of Families of Curves. Initial Conditions

1°. Basic concepts. An equation of the type

$$F(x,\ y,\ y',\ \ldots,\ y^{(n)})=0, \qquad (1)$$

where $y=y(x)$ is the sought-for function, is called a *differential equation of order* n. The function $y=\varphi(x)$, which converts equation (1) into an identity, is called the *solution* of the equation, while the graph of this function is called an *integral curve*. If the solution is represented implicitly, $\Phi(x, y)=0$, then it is usually called an *integral*

Example 1. Check that the function $y=\sin x$ is a solution of the equation

$$y''+y=0.$$

Solution. We have:

$$y'=\cos x, \quad y''=-\sin x$$

and, consequently,

$$y''+y=-\sin x+\sin x\equiv 0.$$

The integral

$$\Phi(x,\ y,\ C_1,\ \ldots,\ C_n)=0 \qquad (2)$$

of the differential equation (1), which contains n independent arbitrary constants $C_1, \ldots, C_n$ and is equivalent (in the given region) to equation (1), is called the *general integral* of this equation (in the respective region). By assigning definite values to the constants $C_1, \ldots, C_n$ in (2), we get *particular integrals*.

Conversely, if we have a family of curves (2) and eliminate the parameters $C_1, \ldots, C_n$ from the system of equations

$$\Phi=0,\ \frac{d\Phi}{dx}=0,\ \ldots,\ \frac{d^n\Phi}{dx^n}=0,$$

we, generally speaking, get a differential equation of type (1) whose general integral in the corresponding region is the relation (2).

Example 2. Find the differential equation of the family of parabolas

$$y=C_1(x-C_2)^2. \qquad (3)$$

Solution. Differentiating equation (3) twice, we get:

$$y'=2C_1(x-C_2) \quad \text{and} \quad y''=2C_1. \qquad (4)$$

Eliminating the parameters C_1 and C_2 from equations (3) and (4), we obtain the desired differential equation

$$2yy''=y'^2.$$

It is easy to verify that the function (3) converts this equation into an identity.

2°. **Initial conditions.** If for the desired particular solution $y=y(x)$ of a differential equation

$$y^{(n)}=f(x, y, y', \ldots, y^{(n-1)}) \qquad (5)$$

the *initial conditions*

$$y(x_0)=y_0, \quad y'(x_0)=y_0', \quad \ldots, \quad y^{(n-1)}(x_0)=y_0^{(n-1)}$$

are given and we know the *general solution* of equation (5)

$$y=\varphi(x, C_1, \ldots, C_n),$$

then the arbitrary constants $C_1, \ldots, C_n$ are determined (if this is possible) from the system of equations

$$\left.\begin{array}{l} y_0=\varphi(x_0, C_1, \ldots, C_n), \\ y_0'=\varphi_x'(x_0, C_1, \ldots, C_n), \\ \cdots\cdots\cdots\cdots \\ y_0^{(n-1)}=\varphi_{x^{n-1}}^{(n-1)}(x_0, C_1, \ldots, C_n). \end{array}\right\}$$

Example 3. Find the curve of the family

$$y=C_1e^x+C_2e^{-2x}, \qquad (6)$$

for which $y(0)=1$, $y'(0)=-2$.

Solution. We have:

$$y'=C_1e^x-2C_2e^{-2x} \qquad (7)$$

Putting $x=0$ in formulas (6) and (7), we obtain

$$1=C_1+C_2, \quad -2=C_1-2C_2,$$

whence

$$C_1=0, \quad C_2=1$$

and, hence,

$$y=e^{-2x}.$$

Determine whether the indicated functions are solutions of the given differential equations:

2704. $xy'=2y$, $y=5x^2$.

2705. $y'^2=x^2+y^2$, $y=\frac{1}{x}$.

2706. $(x+y)\,dx+x\,dy=0$, $y=\frac{C^2-x^2}{2x}$.

2707. $y''+y=0$, $y=3\sin x-4\cos x$.

2708. $\frac{d^2x}{dt^2}+\omega^2x=0$, $x=C_1\cos\omega t+C_2\sin\omega t$.

2709. $y''-2y'+y=0$; a) $y=xe^x$, b) $y=x^2e^x$.

2710. $y''-(\lambda_1+\lambda_2)y'+\lambda_1\lambda_2y=0$,
$y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$.

Show that for the given differential equations the indicated relations are integrals:

2711. $(x-2y)y'=2x-y$, $x^2-xy+y^2=C^2$.

2712. $(x-y+1)y'=1, \quad y=x+Ce^{y}$.
2713. $(xy-x)y''+xy'^2+yy'-2y'=0, \quad y=\ln(xy)$.

Form differential equations of the given families of curves (C, C_1, C_2, C_3 are arbitrary constants):

2714. $y=Cx$.
2715. $y=Cx^2$.
2716. $y^2=2Cx$.
2717. $x^2+y^2=C^2$.
2718. $y=Ce^{x}$.
2719. $x^3=C(x^2-y^2)$.
2720. $y^2+\frac{1}{x}=2+Ce^{-\frac{y^2}{2}}$.
2721. $\ln\frac{x}{y}=1+ay$ (a is a parameter).
2722. $(y-y_0)^2=2px$ (y_0, p are parameters).
2723. $y=C_1e^{2x}+C_2e^{-x}$.
2724. $y=C_1\cos 2x+C_2\sin 2x$.
2725. $y=(C_1+C_2x)e^{x}+C_3$.

2726. Form the differential equation of all straight lines in the xy-plane.

2727. Form the differential equation of all parabolas with vertical axis in the xy-plane.

2728. Form the differential equation of all circles in the xy-plane.

For the given families of curves find the lines that satisfy the given initial conditions:

2729. $x^2-y^2=C, \quad y(0)=5$.
2730. $y=(C_1+C_2x)e^{2x}, \quad y(0)=0, \quad y'(0)=1$.
2731. $y=C_1\sin(x-C_2), \quad y(\pi)=1, \quad y'(\pi)=0$.
2732. $y=C_1e^{-x}+C_2e^{x}+C_3e^{2x}$;
$y(0)=0, \quad y'(0)=1, \quad y''(0)=-2$.

Sec. 2. First-Order Differential Equations

1°. Types of first-order differential equations. A differential equation of the first order in an unknown function y, solved for the derivative y', is of the form

$$y'=f(x, y), \tag{1}$$

where $f(x, y)$ is the given function. In certain cases it is convenient to consider the variable x as the sought-for function, and to write (1) in the form

$$x'=g(x, y), \tag{1'}$$

where $g(x, y)=\frac{1}{f(x, y)}$.

Taking into account that $y'=\frac{dy}{dx}$ and $x'=\frac{dx}{dy}$, the differential equations (1) and (1') may be written in the symmetric form

$$P(x, y)\,dx+Q(x, y)\,dy=0, \tag{2}$$

where $P(x, y)$ and $Q(x, y)$ are known functions.

By solutions to (2) we mean functions of the form $y=\varphi(x)$ or $x=\psi(y)$ that satisfy this equation. The general integral of equations (1) and (1'), or equation (2), is of the form

$$\Phi(x, y, C)=0,$$

where C is an arbitrary constant.

2°. **Direction field.** The set of directions

$$\tan\alpha=f(x, y)$$

is called a direction field of the differential equation (1) and is ordinarily depicted by means of short lines or arrows inclined at an angle α.

Curves $f(x, y)=k$, at the points of which the inclination of the field has a constant value, equal to k, are called *isoclines*. By constructing the isoclines and direction field, it is possible, in the simplest cases, to give a

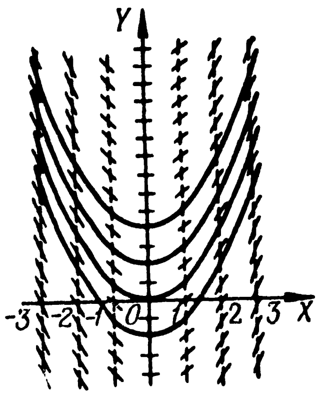

Fig. 105

rough sketch of the field of integral curves, regarding the latter as curves which at each point have the given direction of the field.

Example 1. Using the method of isoclines, construct the field of integral curves of the equation

$$y'=x.$$

Solution. By constructing the isoclines $x=k$ (straight lines) and the direction field, we obtain approximately the field of integral curves (Fig. 105). The family of parabolas

$$y=\frac{x^2}{2}+C$$

is the general solution.

Using the method of isoclines, make approximate constructions of fields of integral curves for the indicated differential equations:

2733. $y'=-x.$

2734. $y'=-\frac{x}{y}.$

2735. $y'=1+y^2.$

2736. $y'=\frac{x+y}{x-y}.$

2737. $y'=x^2+y^2.$

3°. Cauchy's theorem. If a function $f(x, y)$ is continuous in some region $U\{a<x<A,\ b<y<B\}$ and in this region has a bounded derivative $f'_y(x, y)$, then through each point (x_0, y_0) that belongs to U there passes one and only one integral curve $y=\varphi(x)$ of the equation (1) $[\varphi(x_0)=y_0]$.

4°. Euler's broken-line method. For an approximate construction of the integral curve of equation (1) passing through a given point $M_0(x_0, y_0)$, we replace the curve by a broken line with vertices $M_i(x_i, y_i)$, where

$$x_{i+1}=x_i+\Delta x_i,\quad y_{i+1}=y_i+\Delta y_i,$$
$$\Delta x_i=h \quad \text{(one step of the process)},$$
$$\Delta y_i=hf(x_i, y_i) \quad (i=0, 1, 2, \ldots).$$

Example 2. Using Euler's method for the equation

$$y'=\frac{xy}{2},$$

find $y(1)$, if $y(0)=1$ $(h=0.1)$.
We construct the table:

i	x_i	y_i	$\Delta y_i=\frac{x_i y_i}{20}$
0	0	1	0
1	0.1	1	0 005
2	0.2	1.005	0.010
3	0.3	1 015	0 015
4	0 4	1.030	0 021
5	0.5	1.051	0 026
6	0.6	1.077	0 032
7	0.7	1.109	0.039
8	0.8	1.148	0 046
9	0.9	1.194	0.054
10	1.0	1.248	

Thus, $y(1)=1.248$. For the sake of comparison, the exact value is $y(1)=e^{\frac{1}{4}}\approx 1.284$

Using Euler's method, find the particular solutions to the given differential equations for the indicated values of x:

2738. $y'=y$, $y(0)=1$; find $y(1)$ $(h=0.1)$.

2739. $y'=x+y$, $y(1)=1$; find $y(2)$, $(h=0.1)$.

2740. $y'=-\frac{y}{1+x}$, $y(0)=2$; find $y(1)$ $(h=0.1)$.

2741. $y'=y-\frac{2x}{y}$, $y(0)=1$; find $y(1)$ $(h=0.2)$.

Sec. 3. First-Order Differential Equations with Variables Separable. Orthogonal Trajectories

1°. First-order equations with variables separable. An equation with *variables separable* is a first-order equation of the type

$$y' = f(x) g(y) \tag{1}$$

or

$$X(x) Y(y) dx + X_1(x) Y_1(y) dy = 0 \tag{1'}$$

Dividing both sides of equation (1) by $g(y)$ and multiplying by dx, we get $\frac{dy}{g(y)} = f(x) dx$ Whence, by integrating, we get the general integral of equation (1) in the form

$$\int \frac{dy}{g(y)} = \int f(x) dx + C \tag{2}$$

Similarly, dividing both sides of equation (1') by $X_1(x) Y(y)$ and integrating, we get the general integral of (1') in the form

$$\int \frac{X(x)}{X_1(x)} dx + \int \frac{Y_1(y)}{Y(y)} dy = C \tag{2'}$$

If for some value $y = y_0$ we have $g(y_0) = 0$, then the function $y = y_0$ is also (as is directly evident) a solution of equation (1) Similarly, the straight lines $x = a$ and $y = b$ will be the integral curves of equation (1'), if a and b are, respectively, the roots of the equations $X_1(x) = 0$ and $Y(y) = 0$, by the left sides of which we had to divide the initial equation.

Example 1. Solve the equation

$$y' = -\frac{y}{x}. \tag{3}$$

In particular, find the solution that satisfies the initial conditions

$$y(1) = 2$$

Solution. Equation (3) may be written in the form

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Whence, separating variables, we have

$$\frac{dy}{y} = -\frac{dx}{x}$$

and, consequently,

$$\ln|y| = -\ln|x| + \ln C_1,$$

where the arbitrary constant $\ln C_1$ is taken in logarithmic form. After taking antilogarithms we get the general solution

$$y = \frac{C}{x}, \tag{4}$$

where $C = \pm C_1$.

When dividing by y we could lose the solution $y = 0$, but the latter is contained in the formula (4) for $C = 0$.

Utilizing the given initial conditions, we get $C=2$; and, hence, the desired particular solution is

$$y=\frac{2}{x}.$$

2° Certain differential equations that reduce to equations with variables separable. Differential equations of the form

$$y'=f(ax+by+c) \quad (b \neq 0)$$

reduce to equations of the form (1) by means of the substitution $u=ax+by+c$, where u is the new sought-for function

3° Orthogonal trajectories are curves that intersect the lines of the given family $\Phi(x, y, a)=0$ (a is a parameter) at a right angle. If $F(x, y, y')=0$ is the differential equation of the family, then

$$F\left(x, y, -\frac{1}{y'}\right)=0$$

is the differential equation of the orthogonal trajectories.

Example 2. Find the orthogonal trajectories of the family of ellipses

$$x^2+2y^2=a^2. \qquad (5)$$

Solution Differentiating the equation (5), we find the differential equation of the family

$$x+2yy'=0.$$

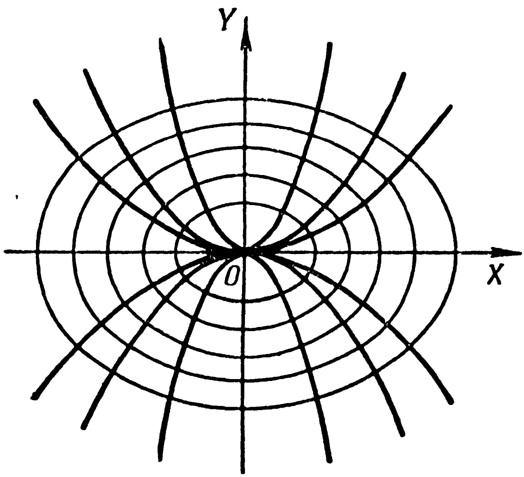

Fig. 106

Whence, replacing y' by $-\frac{1}{y'}$, we get the differential equation of the orthogonal trajectories

$$x-\frac{2y}{y'}=0 \text{ or } y'=\frac{2y}{x}.$$

Integrating, we have $y=Cx^2$ (family of parabolas) (Fig. 106).

4°. Forming differential equations. When forming differential equations in geometrical problems, we can frequently make use of the geometrical meaning of the derivative as the tangent of an angle formed by the tangent line to the curve in the positive x-direction. In many cases this makes it possible straightway to establish a relationship between the ordinate y of the desired curve, its abscissa x, and the tangent of the angle of the tangent line y', that is to say, to obtain the differential equation. In other instances (see Problems 2783, 2890, 2895), use is made of the geometrical significance of the definite integral as the area of a curvilinear trapezoid or the length of an arc. In this case, by hypothesis we have a simple integral equation (since the desired function is under the sign of the integral); however, we can readily pass to a differential equation by differentiating both sides.

Example 3. Find a curve passing through the point (3,2) for which the segment of any tangent line contained between the coordinate axes is divided in half at the point of tangency.

Solution. Let $M(x,y)$ be the mid-point of the tangent line AB, which by hypothesis is the point of tangency (the points A and B are points of intersection of the tangent line with the y- and x-axes). It is given that $OA=2y$ and $OB=2x$. The slope of the tangent to the curve at $M(x, y)$ is

$$\frac{dy}{dx}=-\frac{OA}{OB}=-\frac{y}{x}.$$

This is the differential equation of the sought-for curve. Transforming, we get

$$\frac{dx}{x}+\frac{dy}{y}=0$$

and, consequently,

$$\ln x+\ln y=\ln C \text{ or } xy=C.$$

Utilizing the initial condition, we determine $C=3\cdot 2=6$. Hence, the desired curve is the hyperbola $xy=6$.

Solve the differential equations:

2742. $\tan x\sin^2 y\,dx+\cos^2 x\cot y\,dy=0$.

2743. $xy'-y=y^3$.

2744. $xyy'=1-x^2$.

2745. $y-xy'=a(1+x^2y')$.

2746. $3e^x\tan y\,dx+(1-e^x)\sec^2 y\,dy=0$.

2747. $y'\tan x=y$.

Find the particular solutions of equations that satisfy the indicated initial conditions:

2748. $(1+e^x)\,y\,y'=e^x$; $y=1$ when $x=0$.

2749. $(xy^2+x)\,dx+(x^2y-y)\,dy=0$; $y=1$ when $x=0$.

2750. $y'\sin x=y\ln y$; $y=1$ when $x=\frac{\pi}{2}$.

Solve the differential equations by changing the variables:

2751. $y'=(x+y)^2$.

2752. $y=(8x+2y+1)^2$.

2753. $(2x+3y-1)\,dx+(4x+6y-5)\,dy=0$.

2754. $(2x-y)\,dx+(4x-2y+3)\,dy=0$.

In Examples 2755 and 2756, pass to polar coordinates:

2755. $y'=\frac{\sqrt{x^2+y^2}-x}{y}$.

2756. $(x^2+y^2)\,dx-xy\,dy=0$.

2757*. Find a curve whose segment of the tangent is equal to the distance of the point of tangency from the origin.

2758. Find the curve whose segment of the normal at any point of a curve lying between the coordinate axes is divided in two at this point.

2759. Find a curve whose subtangent is of constant length a.

2760. Find a curve which has a subtangent twice the abscissa of the point of tangency.

2761*. Find a curve whose abscissa of the centre of gravity of an area bounded by the coordinate axes, by this curve and the ordinate of any of its points is equal to 3/4 the abscissa of this point.

2762. Find the equation of a curve that passes through the point (3,1), for which the segment of the tangent between the point of tangency and the x-axis is divided in half at the point of intersection with the y-axis.

2763. Find the equation of a curve which passes through the point (2,0), if the segment of the tangent to the curve between the point of tangency and the y-axis is of constant length 2.

Find the orthogonal trajectories of the given families of curves (a is a parameter), construct the families and their orthogonal trajectories.

2764. $x^2+y^2=a^2$.

2765. $y^2=ax$.

2766. $xy=a$.

2767. $(x-a)^2+y^2=a^2$.

Sec. 4. First-Order Homogeneous Differential Equations

1°. Homogeneous equations. A differential equation

$$P(x, y)\,dx+Q(x, y)\,dy=0 \qquad (1)$$

is called *homogeneous*, if $P(x, y)$ and $Q(x, y)$ are homogeneous functions of the same degree. Equation (1) may be reduced to the form

$$y'=f\left(\frac{y}{x}\right);$$

and by means of the substitution $y=xu$, where u is a new unknown function, it is transformed to an equation with variables separable. We can also apply the substitution $x=yu$.

Example 1. Find the general solution to the equation

$$y'=e^{\frac{y}{x}}+\frac{y}{x}.$$

Solution. Put $y=ux$; then $u+xu'=e^u+u$ or

$$e^{-u}du=\frac{dx}{x}.$$

Integrating, we get $u=-\ln\ln\frac{C}{x}$, whence

$$y=-x\ln\ln\frac{C}{x}.$$

2°. Equations that reduce to homogeneous equations. If

$$y'=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right) \qquad (2)$$

and $\delta=\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}\neq 0$, then, putting into equation (2) $x=u+\alpha$, $y=v+\beta$, where the constants α and β are found from the following system of equations,

$$a_1\alpha+b_1\beta+c_1=0, \quad a_2\alpha+b_2\beta+c_2=0,$$

we get a homogeneous differential equation in the variables u and v. If $\delta=0$, then, putting in (2) $a_1x+b_1y=u$, we get an equation with variables separable.

Integrate the differential equations:

2768. $y'=\frac{y}{x}-1.$

2769. $y'=-\frac{x+y}{x}.$

2770. $(x-y)y\,dx-x^2dy=0.$

2771. For the equation $(x^2+y^2)\,dx-2xy\,dy=0$ find the family of integral curves, and also indicate the curves that pass through the points (4,0) and (1,1), respectively.

2772. $y\,dx+(2\sqrt{xy}-x)\,dy=0.$

2773. $x\,dy-y\,dx=\sqrt{x^2+y^2}dx.$

2774. $(4x^2+3xy+y^2)\,dx+(4y^2+3xy+x^2)\,dy=0.$

2775. Find the particular solution of the equation $(x^2-3y^2)\,dx+2xy\,dy=0$, provided that $y=1$ when $x=2$.

Solve the equations:

2776. $(2x-y+4)\,dy+(x-2y+5)\,dx=0.$

2777. $y'=\frac{1-3x-3y}{1+x+y}.$

2778. $y'=\frac{x+2y+1}{2x+4y+3}.$

2779. Find the equation of a curve that passes through the point (1,0) and has the property that the segment cut off by the tangent line on the y-axis is equal to the radius vector of the point of tangency.

2780**. What shape should the reflector of a search light have so that the rays from a point source of light are reflected as a parallel beam?

2781. Find the equation of a curve whose subtangent is equal to the arithmetic mean of the coordinates of the point of tangency.

2782. Find the equation of a curve for which the segment cut off on the y-axis by the normal at any point of the curve is equal to the distance of this point from the origin.

2783*. Find the equation of a curve for which the area contained between the x-axis, the curve and two ordinates, one of which is a constant and the other a variable, is equal to the ratio of the cube of the variable ordinate to the appropriate abscissa.

2784. Find a curve for which the segment on the y-axis cut off by any tangent line is equal to the abscissa of the point of tangency.

Sec. 5. First-Order Linear Differential Equations. Bernoulli's Equation

1°. Linear equations. A differential equation of the form

$$y' + P(x) \cdot y = Q(x) \tag{1}$$

of degree one in y and y' is called *linear*.

If a function $Q(x) \equiv 0$, then equation (1) takes the form

$$y' + P(x) \cdot y = 0 \tag{2}$$

and is called a *homogeneous linear* differential equation. In this case, the variables may be separated, and we get the general solution of (2) in the form

$$y = C \cdot e^{-\int P(x)\,dx}. \tag{3}$$

To solve the inhomogeneous linear equation (1), we apply a method that is called *variation of parameters*, which consists in first finding the general solution of the respective homogeneous linear equation, that is, relationship (3). Then, assuming here that C is a function of x, we seek the solution of the inhomogeneous equation (1) in the form of (3). To do this, we put into (1) y and y' which are found from (3), and then from the differential equation thus obtained we determine the function $C(x)$. We thus get the general solution of the inhomogeneous equation (1) in the form

$$y = C(x) \cdot e^{-\int P(x)\,dx}.$$

Example 1. Solve the equation

$$y' = \tan x \cdot y + \cos x. \tag{4}$$

Solution. The corresponding homogeneous equation is

$$y' - \tan x \cdot y = 0.$$

Solving it we get:

$$y = C \cdot \frac{1}{\cos x}.$$

Considering C as a function of x, and differentiating, we find:

$$y=\frac{1}{\cos x}\cdot\frac{dC}{dx}+\frac{\sin x}{\cos^2 x}\cdot C.$$

Putting y and y' into (4), we get:

$$\frac{1}{\cos x}\cdot\frac{dC}{dx}+\frac{\sin x}{\cos^2 x}\cdot C=\tan x\cdot\frac{C}{\cos x}+\cos x,\quad\text{or}\quad\frac{dC}{dx}=\cos^2 x,$$

whence

$$C(x)=\int\cos^2 x\,dx=\frac{1}{2}x+\frac{1}{4}\sin 2x+C_1.$$

Hence, the general solution of equation (4) has the form

$$y=\left(\frac{1}{2}x+\frac{1}{4}\sin 2x+C_1\right)\cdot\frac{1}{\cos x}.$$

In solving the linear equation (1) we can also make use of the substitution

$$y=uv, \tag{5}$$

where u and v are functions of x. Then equation (1) will have the form

$$[u'+P(x)u]v+v'u=Q(x). \tag{6}$$

If we require that

$$u'+P(x)u=0, \tag{7}$$

then from (7) we find u, and from (6) we find v; hence, from (5) we find y.

2°. Bernoulli's equation. A first-order equation of the form

$$y'+P(x)y=Q(x)y^{\alpha},$$

where $\alpha\neq 0$ and $\alpha\neq 1$, is called *Bernoulli's equation*. It is reduced to a linear equation by means of the substitution $z=y^{1-\alpha}$. It is also possible to apply directly the substitution $y=uv$, or the method of variation of parameters.

Example 2. Solve the equation

$$y'=\frac{4}{x}y+x\sqrt{y}.$$

Solution. This is Bernoulli's equation. Putting

$$y=u\cdot v,$$

we get

$$u'v+v'u=\frac{4}{x}uv+x\sqrt{uv}\quad\text{or}\quad v\left(u'-\frac{4}{x}u\right)+v'u=x\sqrt{uv}. \tag{8}$$

To determine the function u we require that the relation

$$u'-\frac{4}{x}u=0$$

be fulfilled, whence we have

$$u=x^4.$$

Putting this expression into (8), we get

$$v'x^4=x\sqrt{vx^4},$$

whence we find v:

$$v=\left(\frac{1}{2}\ln x+c\right)^2,$$

and, consequently, the general solution is obtained in the form

$$y=x^4\left(\frac{1}{2}\ln x+C\right)^2.$$

Find the general integrals of the equations:

2785. $\frac{dy}{dx}-\frac{y}{x}=x$.

2786. $\frac{dy}{dx}+\frac{2y}{x}=x^3$.

2787*. $(1+y^2)\,dx=(\sqrt{1+y^2}\sin y-xy)\,dy$.

2788. $y^2dx-(2xy+3)\,dy=0$.

Find the particular solutions that satisfy the indicated conditions:

2789. $xy'+y-e^x=0$; $y=b$ when $x=a$.

2790. $y'-\frac{y}{1-x^2}-1-x=0$; $y=0$ when $x=0$.

2791. $y'-y\tan x=\frac{1}{\cos x}$; $y=0$ when $x=0$.

Find the general solutions of the equations:

2792. $\frac{dy}{dx}+\frac{y}{x}=-xy^2$.

2793. $2xy\frac{dy}{dx}-y^2+x=0$.

2794. $y\,dx+\left(x-\frac{1}{2}x^3y\right)dy=0$.

2795. $3x\,dy=y(1+x\sin x-3y^3\sin x)\,dx$.

2796. Given three particular solutions y, y_1, y_2 of a linear equation. Prove that the expression $\frac{y_2-y}{y-y_1}$ remains unchanged for any x. What is the geometrical significance of this result?

2797. Find the curves for which the area of a triangle formed by the x-axis, a tangent line and the radius vector of the point of tangency is constant.

2798. Find the equation of a curve, a segment of which, cut off on the x-axis by a tangent line, is equal to the square of the ordinate of the point of tangency.

2799. Find the equation of a curve, a segment of which, cut off on the y-axis by a tangent line, is equal to the subnormal.

2800. Find the equation of a curve, a segment of which, cut off on the y-axis by a tangent line, is proportional to the square of the ordinate of the point of tangency.

2801. Find the equation of the curve for which the segment of the tangent is equal to the distance of the point of intersection of this tangent with the x-axis from the point $M(0,a)$.

Sec. 6. Exact Differential Equations. Integrating Factor

1°. **Exact differential equations.** If for the differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0 \tag{1}$$

the equality $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is fulfilled, then equation (1) may be written in the form $dU(x, y) = 0$ and is then called an *exact differential equation*. The general integral of equation (1) is $U(x, y) = C$. The function $U(x, y)$ is determined by the technique given in Ch. VI, Sec. 8, or from the formula

$$U = \int_{x_0}^{x} P(x, y)\,dx + \int_{y_0}^{y} Q(x_0, y)\,dy$$

(see Ch. VII, Sec. 9).

Example 1. Find the general integral of the differential equation

$$(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0.$$

Solution. This is an exact differential equation, since $\frac{\partial(3x^2 + 6xy^2)}{\partial y} = \frac{\partial(6x^2y + 4y^3)}{\partial x} = 12xy$ and, hence, the equation is of the form $dU = 0$.

Here,

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy^2 \quad \text{and} \quad \frac{\partial U}{\partial y} = 6x^2y + 4y^3;$$

whence

$$U = \int (3x^2 + 6xy^2)\,dx + \varphi(y) = x^3 + 3x^2y^2 + \varphi(y).$$

Differentiating U with respect to y, we find $\frac{\partial U}{\partial y} = 6x^2y + \varphi'(y) = 6x^2y + 4y^3$ (by hypothesis); from this we get $\varphi'(y) = 4y^3$ and $\varphi(y) = y^4 + C_0$. We finally get $U(x, y) = x^3 + 3x^2y^2 + y^4 + C_0$, consequently, $x^3 + 3x^2y^2 + y^4 = C$ is the sought-for general integral of the equation.

2°. **Integrating factor.** If the left side of equation (1) is not a total (exact) differential and the conditions of the Cauchy theorem are fulfilled, then there exists a function $\mu = \mu(x, y)$ *(integrating factor)* such that

$$\mu(P\,dx + Q\,dy) = dU. \tag{2}$$

Whence it is found that the function μ satisfies the equation

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q).$$

The integrating factor μ is readily found in two cases:

1) $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = F(x)$, then $\mu = \mu(x)$;

2) $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = F_1(y)$, then $\mu = \mu(y)$.

Example 2. Solve the equation $\left(2xy+x^2y+\frac{y^3}{3}\right)dx+(x^2+y^2)\,dy=0$.

Solution. Here $P=2xy+x^2y+\frac{y^3}{3}$, $Q=x^2+y^2$

and $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{2x+x^2+y^2-2x}{x^2+y^2}=1$, hence, $\mu=\mu(x)$.

Since $\frac{\partial(\mu P)}{\partial y}=\frac{\partial(\mu Q)}{\partial x}$ or $\mu\frac{\partial P}{\partial y}=\mu\frac{\partial Q}{\partial x}+Q\frac{d\mu}{dx}$,

it follows that

$$\frac{d\mu}{\mu}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)dx=dx \text{ and } \ln\mu=x,\ \mu=e^x.$$

Multiplying the equation by $\mu=e^x$, we obtain

$$e^x\left(2xy+x^2y+\frac{y^3}{3}\right)dx+e^x(x^2+y^2)\,dy=0$$

which is an exact differential equation. Integrating it, we get the general integral

$$ye^x\left(x^2+\frac{y^2}{3}\right)=C.$$

Find the general integrals of the equations:

2802 $(x+y)\,dx+(x+2y)\,dy=0$.

2803. $(x^2+y^2+2x)\,dx+2xy\,dy=0$.

2804. $(x^3-3xy^2+2)\,dx-(3x^2y-y^2)\,dy=0$.

2805. $x\,dx-y\,dy=\frac{x\,dy-y\,dx}{x^2+y^2}$.

2806. $\frac{2x\,dx}{y^3}+\frac{y^2-3x^2}{y^4}\,dy=0$.

2807. Find the particular integral of the equation

$$\left(x+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0,$$

which satisfies the initial condition $y(0)=2$.

Solve the equations that admit of an integrating factor of the form $\mu=\mu(x)$ or $\mu=\mu(y)$:

2808. $(x+y^2)\,dx-2xy\,dy=0$.

2809. $y(1+xy)\,dx-x\,dy=0$.

2810. $\frac{y}{x}dx+(y^3-\ln x)dy=0$.

2811. $(x\cos y-y\sin y)\,dy+(x\sin y+y\cos y)\,dx=0$.

Sec. 7. First-Order Differential Equations not Solved for the Derivative

1°. First-order differential equations of higher powers. If an equation

$$F(x, y, y') = 0, \tag{1}$$

which for example is of degree two in y', then by solving (1) for y' we get two equations:

$$y' = f_1(x, y), \quad y' = f_2(x, y). \tag{2}$$

Thus, generally speaking, through each point $M_0(x_0, y_0)$ of some region of a plane there pass two integral curves. The general integral of equation (1) then, generally speaking, has the form

$$\Phi(x, y, C) \equiv \Phi_1(x, y, C)\,\Phi_2(x, y, C) = 0, \tag{3}$$

where Φ_1 and Φ_2 are the general integrals of equations (2).

Besides, there may be a *singular integral* for equation (1). Geometrically, a singular integral is the envelope of a family of curves (3) and may be obtained by eliminating C from the system of equations

$$\Phi(x, y, C) = 0, \quad \Phi'_C(x, y, C) = 0 \tag{4}$$

or by eliminating $p = y'$ from the system of equations

$$F(x, y, p) = 0, \quad F'_p(x, y, p) = 0. \tag{5}$$

We note that the curves defined by the equations (4) or (5) are not always solutions of equation (1); therefore, in each case, a check is necessary.

Example 1. Find the general and singular integrals of the equation

$$xy'^2 + 2xy' - y = 0.$$

Solution. Solving for y' we have two homogeneous equations:

$$y' = -1 + \sqrt{1 + \frac{y}{x}}, \quad y' = -1 - \sqrt{1 + \frac{y}{x}},$$

defined in the region

$$x(x + y) > 0,$$

the general integrals of which are

$$\left(\sqrt{1 + \frac{y}{x}} - 1\right)^2 = \frac{C}{x}, \qquad \left(\sqrt{1 + \frac{y}{x}} + 1\right)^2 = \frac{C}{x}$$

or

$$(2x + y - C) - 2\sqrt{x^2 + xy} = 0, \qquad (2x + y - C) + 2\sqrt{x^2 + xy} = 0.$$

Multiplying, we get the general integral of the given equation

$$(2x + y - C)^2 - 4(x^2 + xy) = 0$$

or

$$(y - C)^2 = 4Cx$$

(a family of parabolas).

Differentiating the general integral with respect to C and eliminating C, we find the singular integral

$$y + x = 0.$$

(It may be verified that $y + x = 0$ is the solution of this equation.)

It is also possible to find the singular integral by differentiating $xp^2+2xp-y=0$ with respect to p and eliminating p.

2°. Solving a differential equation by introducing a parameter. If a first-order differential equation is of the form

$$x=\varphi(y,\ y'),$$

then the variables y and x may be determined from the system of equations

$$\frac{1}{p}=\frac{\partial\varphi}{\partial y}+\frac{\partial\varphi}{\partial}\frac{dp}{dy}, \quad x=\varphi(y,\ p),$$

where $p=y'$ plays the part of a parameter.

Similarly, if $y=\psi(x,\ y')$, then x and y are determined from the system of equations

$$p=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial p}\frac{dp}{dx}, \qquad y=\psi(x,\ p).$$

Example 2. Find the general and singular integrals of the equation

$$y=y'^2-xy'+\frac{x^2}{2}.$$

Solution. Making the substitution $y'=p$, we rewrite the equation in the form

$$y=p^2-xp+\frac{x^2}{2}.$$

Differentiating with respect to x and considering p a function of x, we have

$$p=2p\frac{dp}{dx}-p-x\frac{dp}{dx}+x$$

or $\frac{dp}{dx}(2p-x)=(2p-x)$, or $\frac{dp}{dx}=1$. Integrating we get $p=x+C$. Substituting into the original equation, we have the general solution

$$y=(x+C)^2-x(x+C)+\frac{x^2}{2} \text{ or } y=\frac{x^2}{2}+Cx+C^2.$$

Differentiating the general solution with respect to C and eliminating C, we obtain the singular solution: $y=\frac{x^2}{4}$. (It may be verified that $y=\frac{x^2}{4}$ is the solution of the given equation.)

If we equate to zero the factor $2p-x$, which was cancelled out, we get $p=\frac{x}{2}$ and, putting p into the given equation, we get $y=\frac{x^2}{4}$, which is the same singular solution.

Find the general and singular integrals of the equations: (In Problems 2812 and 2813 construct the field of integral curves.)

2812. $y'^2-\frac{2y}{x}y'+1=0.$

2813. $4y'^2-9x=0.$

2814. $yy'^2-(xy+1)y'+x=0$.

2815. $yy'^2-2xy'+y=0$.

2816. Find the integral curves of the equation $y'^2+y^2=1$ that pass through the point $M\left(0, \frac{1}{2}\right)$.

Introducing the parameter $y'=p$, solve the equations:

2817. $x=\sin y'+\ln y'$.

2818. $y=y'^2e^{y'}$.

2819. $y=y'^2+2\ln y'$.

2820. $4y=x^2+y'^2$.

2821. $e^x=\frac{y^2+y'^2}{2y'}$.

Sec. 8. The Lagrange and Clairaut Equaticns

1°. Lagrange's equation. An equation of the form

$$y=x\varphi(p)+\psi(p), \tag{1}$$

where $p=y'$ is called *Lagrange's equation* Equation (1) is reduced to a linear equation in x by differentiation and taking into consideration that $dy=p\,dx$:

$$p\,dx=\varphi(p)\,dx+[x\varphi'(p)+\psi'(p)]\,dp. \tag{2}$$

If $p\not\equiv\varphi(p)$, then from (1) and (2) we get the general solution in parametric form:

$$x=Cf(p)+g(p),\quad y=[Cf(p)+g(p)]\,\varphi(p)+\psi(p),$$

where p is a parameter and $f(p)$, $g(p)$ are certain known functions. Besides, there may be a singular solution that is found in the usual way.

2°. Clairaut's equation. If in equation (1) $p\equiv\varphi(p)$, then we get *Clairaut's equation*

$$y=xp+\psi(p).$$

Its general solution is of the form $y=Cx+\psi(C)$ (a family of straight lines). There is also a *particular solution* (envelope) that results by eliminating the parameter p from the system of equations

$$\begin{cases} x=-\psi'(p), \\ y=px+\psi(p). \end{cases}$$

Example. Solve the equation

$$y=2y'x+\frac{1}{y'}. \tag{3}$$

Solution. Putting $y'=p$ we have $y=2px+\frac{1}{p}$; differentiating and replacing dy by $p\,dx$, we get

$$p\,dx=2p\,dx+2x\,dp-\frac{dp}{p^2}$$

or

$$\frac{dx}{dp}=-\frac{2}{p}x+\frac{1}{p^3}.$$

Solving this linear equation, we will have

$$x=\frac{1}{p^2}(\ln p+C).$$

Hence, the general integral will be

$$\begin{cases} x=\dfrac{1}{p^2}(\ln p+C), \\ y=2px+\dfrac{1}{p}. \end{cases}$$

To find the singular integral, we form the system

$$y=2px+\frac{1}{p}, \quad 0=2x-\frac{1}{p^2}$$

in the usual way. Whence

$$x=\frac{1}{2p^2}, \quad y=\frac{2}{p}$$

and, consequently,

$$y=\pm 2\sqrt{2x}.$$

Putting y into (3) we are convinced that the function obtained is not a solution and, therefore, equation (3) does not have a singular integral.

Solve the Lagrange equations:

2822. $y=\frac{1}{2}x\left(y'+\frac{y}{y'}\right)$.

2823. $y=y'+\sqrt{1-y'^2}$.

2824. $y=(1+y')x+y'^2$.

2825*. $y=-\frac{1}{2}y'(2x+y')$.

Find the general and singular integrals of the Clairaut equations and construct the field of integral curves:

2826. $y=xy'+y'^2$.

2827. $y=xy'+y'$.

2828. $y=xy'+\sqrt{1+(y')^2}$.

2829. $y=xy'+\frac{1}{y'}$.

2830. Find the curve for which the area of a triangle formed by a tangent at any point and by the coordinate axes is constant.

2831. Find the curve if the distance of a given point to any tangent to this curve is constant.

2832. Find the curve for which the segment of any of its tangents lying between the coordinate axes has constant length l.

Sec. 9. Miscellaneous Exercises on First-Order Differential Equations

2833. Determine the types of differential equations and indicate methods for their solution:

a) $(x+y)y'=x\arctan\frac{y}{x}$;

b) $(x-y)y'=y^2$;

c) $y'=2xy+x^3$;

d) $y'=2xy+y^3$;

e) $xy'+y=\sin y$;

f) $(y-xy')^2=y'^3$;

g) $y=xe^{y'}$;

h) $(y'-2xy)\sqrt{y}=x^3$;

i) $y' = (x + y)^2$;
j) $x \cos y' + y \sin y' = 1$;
k) $(x^2 - xy) y' = y^4$;
l) $(x^2 + 2xy^3)\,dx + (y^2 + 3x^2y^2)\,dy = 0$;
m) $(x^3 - 3xy)\,dx + (x^2 + 3)\,dy = 0$;
n) $(xy^3 + \ln x)\,dx = y^2 dy$.

Solve the equations:

2834. a) $\left(x - y \cos \frac{y}{x}\right) dx + x \cos \frac{y}{x}\,dy = 0$;

b) $x \ln \frac{x}{y}\,dy - y\,dx = 0$.

2835. $x\,dx = \left(\frac{x^2}{y} - y^3\right) dy$.

2836. $(2xy^2 - y)\,dx + x\,dy = 0$.

2837. $xy' + y = xy^2 \ln x$.

2838. $y = xy' + y' \ln y'$.

2839. $y = xy' + \sqrt{-ay'}$.

2840. $x^2 (y + 1)\,dx + (x^3 - 1)(y - 1)\,dy = 0$.

2841. $(1 + y^2)(e^{2x}\,dx - e^y\,dy) - (1 + y)\,dy = 0$.

2842. $y' - y \frac{2x - 1}{x^2} = 1$.

2843. $ye^y = (y^3 + 2xe^y)\,y'$.

2844. $y' + y \cos x = \sin x \cos x$.

2845. $(1 - x^2)\,y' + xy = a$.

2846. $xy' - \frac{y}{x+1} - x = 0$.

2847. $y' (x \cos y + a \sin 2y) = 1$.

2848. $(x^2y - x^2 + y - 1)\,dx + (xy + 2x - 3y - 6)\,dy = 0$.

2849. $y' = \left(1 + \frac{y - 1}{2x}\right)^2$.

2850. $xy^3\,dx = (x^2y + 2)\,dy$.

2851. $y' = \frac{3x^2}{x^3 + y + 1}$.

2852. $2dx + \sqrt{\frac{x}{y}}\,dy - \sqrt{\frac{y}{x}}\,dx = 0$.

2853. $y' = \frac{y}{x} + \tan \frac{y}{x}$.

2854. $yy' + y^2 = \cos x$.

2855. $x\,dy + y\,dx = y^2\,dx$.

2856. $y' (x + \sin y) = 1$.

2857. $y \frac{dp}{dy} = -p + p^2$.

2858. $x^3\,dx - (x^4 + y^3)\,dy = 0$.

2859. $x^2y'^2 + 3xyy' + 2y^2 = 0$.

2860. $\frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} + \frac{x\,dy - y\,dx}{y^2} = 0$.

2861. $e^y\,dx + (xe^y - 2y)\,dy = 0$.

2862. $y = 2xy' + \sqrt{1 + y'^2}$.

2863. $y' = \frac{y}{x}(1 + \ln y - \ln x)$.

2864. $(2e^x + y^4)\,dy - ye^x\,dx = 0$.

2865. $y' = 2\left(\frac{y + 2}{x + y - 1}\right)^2$.

2866. $xy\,(xy^2 + 1)\,dy - dx = 0$.

2867. $a\,(xy' + 2y) = xyy'$.

2868. $x\,dy - y\,dx = y^2\,dx$.

2869. $(x^2-1)^{3/2}\,dy+(x^3+3xy\sqrt{x^2-1})\,dx=0.$

2870. $\tan x\dfrac{dy}{dx}-y=a.$

2871. $\sqrt{a^2+x^2}\,dy+(x+y-\sqrt{a^2+x^2})\,dx=0.$

2872. $xyy'^2-(x^2+y^2)\,y'+xy=0.$

2873. $y=xy'+\dfrac{1}{y'^2}.$

2874. $(3x^2+2xy-y^2)\,dx+(x^2-2xy-3y^2)\,dy=0.$

2875. $2yp\dfrac{dp}{dy}=3p^2+4y^2.$

Find solutions to the equations for the indicated initial conditions:

2876. $y'=\dfrac{y+1}{x}$; $y=0$ for $x=1$.

2877. $e^{x-y}y'=1$; $y=1$ for $x=1$.

2878. $\cot xy'+y=2$; $y=2$ for $x=0$.

2879. $e^y(y'+1)=1$; $y=0$ for $x=0$.

2880. $y'+y=\cos x$; $y=\dfrac{1}{2}$ for $x=0$.

2881. $y'-2y=-x^2$; $y=\dfrac{1}{4}$ for $x=0$.

2882. $y'+y=2x$; $y=-1$ for $x=0$.

2883. $xy'=y$; a) $y=1$ for $x=1$; b) $y=0$ for $x=0$.

2884. $2xy'=y$; a) $y=1$ for $x=1$; b) $y=0$ for $x=0$.

2885. $2xyy'+x^2-y^2=0$; a) $y=0$ for $x=0$; b) $y=1$ for $x=0$; c) $y=0$ for $x=1$.

2886. Find the curve passing through the point (0, 1), for which the subtangent is equal to the sum of the coordinates of the point of tangency.

2887. Find a curve if we know that the sum of the segments cut off on the coordinate axes by a tangent to it is constant and equal to $2a$.

2888. The sum of the lengths of the normal and subnormal is equal to unity. Find the equation of the curve if it is known that the curve passes through the coordinate origin.

2889*. Find a curve whose angle formed by a tangent and the radius vector of the point of tangency is constant.

2890. Find a curve knowing that the area contained between the coordinate axes, this curve and the ordinate of any point on it is equal to the cube of the ordinate.

2891. Find a curve knowing that the area of a sector bounded by the polar axis, by this curve and by the radius vector of any point of it is proportional to the cube of this radius vector.

2892. Find a curve, the segment of which, cut off by the tangent on the x-axis, is equal to the length of the tangent.

2893. Find the curve, of which the segment of the tangent contained between the coordinate axes is divided into half by the parabola $y^2 = 2x$.

2894. Find the curve whose normal at any point of it is equal to the distance of this point from the origin.

2895*. The area bounded by a curve, the coordinate axes, and the ordinate of some point of the curve is equal to the length of the corresponding arc of the curve. Find the equation of this curve if it is known that the latter passes through the point (0, 1).

2896. Find the curve for which the area of a triangle formed by the x-axis, a tangent, and the radius vector of the point of tangency is constant and equal to a^2.

2897. Find the curve if we know that the mid-point of the segment cut off on the x-axis by a tangent and a normal to the curve is a constant point $(a, 0)$.

When forming first-order differential equations, particularly in physical problems, it is frequently advisable to apply the so-called *method of differentials*, which consists in the fact that approximate relationships between infinitesimal increments of the desired quantities (these relationships are accurate to infinitesimals of higher order) are replaced by the corresponding relationships between their differentials. This does not affect the result.

Problem. A tank contains 100 litres of an aqueous solution containing 10 kg of salt. Water is entering the tank at the rate of 3 litres per minute, and the mixture is flowing out at 2 litres per minute. The concentration is maintained uniform by stirring. How much salt will the tank contain at the end of one hour?

Solution. The concentration c of a substance is the quantity of it in unit volume. If the concentration is uniform, then the quantity of substance in volume V is cV.

Let the quantity of salt in the tank at the end of t minutes be x kg. The quantity of solution in the tank at that instant will be $100+t$ litres, and, consequently, the concentration $c=\frac{x}{100+t}$ kg per litre.

During time dt, $2dt$ litres of the solution flows out of the tank (the solution contains $2c\,dt$ kg of salt). Therefore, a change of dx in the quantity of salt in the tank is given by the relationship

$$-dx=2c\,dt=\frac{2x}{100+t}\,dt.$$

This is the sought-for differential equation. Separating variables and integrating, we obtain

$$\ln x=-2\ln(100+t)+\ln C$$

or

$$x=\frac{C}{(100+t)^2}.$$

The constant C is found from the fact that when $t=0$, $x=10$, that is, $C=100{,}000$. At the expiration of one hour, the tank will contain $x=\frac{100{,}000}{160^2}\approx 3.9$ kilograms of salt.

2898*. Prove that for a heavy liquid rotating about a vertical axis the free surface has the form of a paraboloid of revolution.

2899*. Find the relationship between the air pressure and the altitude if it is known that the pressure is 1 kgf on 1 cm² at sea level and 0.92 kgf on 1 cm² at an altitude of 500 metres.

2900*. According to Hooke's law an elastic band of length l increases in length klF $(k=\text{const})$ due to a tensile force F. By how much will the band increase in length due to its weight W if the band is suspended at one end? (The initial length of the band is l.)

2901. Solve the same problem for a weight P suspended from the end of the band.

When solving Problems 2902 and 2903, make use of Newton's law, by which the rate of cooling of a body is proportional to the difference of temperatures of the body and the ambient medium.

2902. Find the relationship between the temperature T and the time t if a body, heated to T_0 degrees, is brought into a room at constant temperature (a degrees).

2903. During what time will a body heated to 100° cool off to 30° if the temperature of the room is 20° and during the first 20 minutes the body cooled to 60°?

2904. The retarding action of friction on a disk rotating in a liquid is proportional to the angular velocity of rotation. Find the relationship between the angular velocity and time if it is known that the disk began rotating at 100 rpm and after one minute was rotating at 60 rpm.

2905*. The rate of disintegration of radium is proportional to the quantity of radium present. Radium disintegrates by one half in 1600 years. Find the percentage of radium that has disintegrated after 100 years.

2906*. The rate of outflow of water from an aperture at a vertical distance h from the free surface is defined by the formula

$$v=c\sqrt{2gh},$$

where $c\approx 0.6$ and g is the acceleration of gravity.

During what period of time will the water filling a hemispherical boiler of diameter 2 metres flow out of it through a circular opening of radius 0.1 m in the bottom.

2907*. The quantity of light absorbed in passing through a thin layer of water is proportional to the quantity of incident light and to the thickness of the layer. If one half of the original quantity of light is absorbed in passing through a three-metre-thick layer of water, what part of this quantity will reach a depth of 30 metres?

2908*. The air resistance to a body falling with a parachute is proportional to the square of the rate of fall. Find the limiting velocity of descent.

2909*. The bottom of a tank with a capacity of 300 litres is covered with a mixture of salt and some insoluble substance. Assuming that the rate at which the salt dissolves is proportional to the difference between the concentration at the given time and the concentration of a saturated solution (1 kg of salt per 3 litres of water) and that the given quantity of pure water dissolves 1/3 kg of salt in 1 minute, find the quantity of salt in solution at the expiration of one hour.

2910*. The electromotive force e in a circuit with current i, resistance R and self-induction L is made up of the voltage drop Ri and the electromotive force of self-induction $L\frac{di}{dt}$. Determine the current i at time t if $e=E\sin\omega t$ (E and ω are constants) and $i=0$ when $t=0$.

Sec. 10. Higher-Order Differential Equations

1°. The case of direct integration. If

$$y^{(n)}=f(x),$$

then

$$y=\underbrace{\int dx\int\ldots\int}_{n\ \text{times}} f(x)\,dx+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_n.$$

2°. Cases of reduction of order. 1) If a differential equation does not contain y explicitly, for instance,

$$F(x,\ y',\ y'')=0,$$

then, assuming $y'=p$, we get an equation of an order one unit lower:

$$F(x,\ p,\ p')=0.$$

Example 1. Find the particular solution of the equation

$$xy''+y'+x=0,$$

that satisfies the conditions

$$y=0,\ y'=0 \text{ when } x=0.$$

Solution. Putting $y'=p$, we have $y''=p'$, whence

$$xp'+p+x=0.$$

Solving the latter equation as a linear equation in the function p, we get

$$px=C_1-\frac{x^2}{2}.$$

From the fact that $y'=p=0$ when $x=0$, we have $0=C_1-0$, i.e., $C_1=0$. Hence,

$$p=-\frac{x}{2}$$

or

$$\frac{dy}{dx}=-\frac{x}{2},$$

whence, integrating once again, we obtain

$$y=-\frac{x^2}{4}+C_2$$

Putting $y=0$ when $x=0$, we find $C_2=0$. Hence, the desired particular solution is

$$y=-\frac{1}{4}x^2.$$

2) If a differential equation does not contain x explicitly, for instance,

$$F(y, y', y'')=0$$

then, putting $y'=p$, $y''=p\frac{dp}{dy}$, we get an equation of an order one unit lower:

$$F\left(y, p, p\frac{dp}{dy}\right)=0.$$

Example 2. Find the particular solution of the equation

$$yy''-y'^2=y^4$$

provided that $y=1$, $y'=0$ when $x=0$.

Solution. Put $y'=p$, then $y''=p\frac{dp}{dy}$ and our equation becomes

$$yp\frac{dp}{dy}-p^2=y^4.$$

We have obtained an equation of the Bernoulli type in p (y is considered the argument). Solving it, we find

$$p=\pm y\sqrt{C_1+y^2}.$$

From the fact that $y'=p=0$ when $y=1$, we have $C_1=-1$. Hence,

$$p=\pm y\sqrt{y^2-1}$$

or

$$\frac{dy}{dx}=\pm y\sqrt{y^2-1}.$$

Integrating, we have

$$\arccos\frac{1}{y}\pm x=C_2.$$

Putting $y=1$ and $x=0$, we obtain $C_2=0$, whence $\frac{1}{y}=\cos x$ or $y=\sec x$.

Solve the following equations:

2911. $y'' = \frac{1}{x}$.

2912. $y'' = -\frac{2}{2y^3}$.

2913. $y'' = 1 - y'^2$.

2914. $xy'' + y' = 0$.

2915. $yy'' = y'^2$.

2916. $yy'' + y'^2 = 0$.

2917. $(1 + x^2) y'' + y'^2 + 1 = 0$.

2918. $y' (1 + y'^2) = ay''$.

2919. $x^2 y'' + xy' = 1$.

2920. $yy'' = y^2 y' + y'^2$.

2921. $yy'' - y' (1 + y') = 0$.

2922. $y'' = -\frac{x}{y'}$.

2923. $(x + 1) y'' - (x + 2) y' + x + 2 = 0$.

2924. $xy'' = y' \ln \frac{y'}{x}$.

2925. $y' + \frac{1}{4} y''^2 = xy''$.

2926. $xy''' + y'' = 1 + x$.

2927. $y'''^2 + y''^2 = 1$.

Find the particular solutions for the indicated initial conditions:

2928. $(1 + x^2) y'' - 2xy' = 0$; $y = 0$, $y' = 3$ for $x = 0$.

2929. $1 + y'^2 = 2yy''$; $y = 1$, $y' = 1$ for $x = 1$.

2930. $yy'' + y'^2 = y'^3$; $y = 1$, $y' = 1$ for $x = 0$.

2931. $xy'' = y'$; $y = 0$, $y' = 0$ for $x = 0$.

Find the general integrals of the following equations:

2932. $yy' = \sqrt{y^2 + y'^2}\, y'' - y'y''$.

2933. $yy'' = y'^2 + y' \sqrt{y^2 + y'^2}$.

2934. $y'^2 - yy'' = y^2 y'$.

2935. $yy'' + y'^2 - y'^3 \ln y = 0$.

Find solutions that satisfy the indicated conditions:

2936. $y'' y^3 = 1$; $y = 1$, $y' = 1$ for $x = \frac{1}{2}$.

2937. $yy'' + y'^2 = 1$; $y = 1$, $y' = 1$ for $x = 0$.

2938. $xy'' = \sqrt{1 + y'^2}$; $y = 0$ for $x = 1$; $y = 1$ for $x = e^2$.

2939. $y'' (1 + \ln x) + \frac{1}{x} \cdot y' = 2 + \ln x$; $y = \frac{1}{2}$, $y' = 1$ for $x = 1$.

2940. $y'' = \frac{y'}{x} \left(1 + \ln \frac{y'}{x}\right)$; $y = \frac{1}{2}$, $y' = 1$ for $x = 1$.

2941. $y'' - y'^2 + y' (y - 1) = 0$; $y = 2$, $y' = 2$ for $x = 0$.

2942. $3y'y'' = y + y'^3 + 1$; $y = -2$, $y' = 0$ for $x = 0$.

2943. $y^2 + y'^2 - 2yy'' = 0$; $y = 1$, $y' = 1$ for $x = 0$.

2944. $yy' + y'^2 + yy'' = 0$; $y = 1$ for $x = 0$ and $y = 0$ for $x = -1$.

2945. $2y'+(y'^2-6x)\cdot y''=0$; $y=0$, $y'=2$ for $x=2$.
2946. $y'y^2+yy''-y'^2=0$; $y=1$, $y'=2$ for $x=0$.
2947. $2yy''-3y'^2=4y^2$; $y=1$, $y'=0$ for $x=0$.
2948. $2yy''+y^2-y'^2=0$; $y=1$, $y'=1$ for $x=0$.
2949. $y''=y'^2-y$; $y=-\frac{1}{4}$, $y'=\frac{1}{2}$ for $x=1$.
2950. $y''+\frac{1}{y^2}e^{y^2}y'-2yy'^2=0$; $y=1$, $y'=e$ for $x=-\frac{1}{2e}$.
2951. $1+yy''+y'^2=0$; $y=0$, $y'=1$ for $x=1$.
2952. $(1+yy')y''=(1+y'^2)y'$; $y=1$, $y'=1$ for $x=0$.
2953. $(x+1)y''+xy'^2=y'$; $y=-2$, $y'=4$ for $x=1$.

Solve the equations:

2954. $y'=xy''^2+y''^2$.
2955. $y'=xy''+y''-y''^2$.
2956. $y'''^2=4y''$.

2957. $yy'y''=y'^3+y''^2$. Choose the integral curve passing through the point (0, 0) and tangent, at it, to the straight line $y+x=0$.

2958. Find the curves of constant radius of curvature.

2959. Find a curve whose radius of curvature is proportional to the cube of the normal.

2960. Find a curve whose radius of curvature is equal to the normal.

2961. Find a curve whose radius of curvature is double the normal.

2962. Find the curves whose projection of the radius of curvature on the y-axis is a constant.

2963. Find the equation of the cable of a suspension bridge on the assumption that the load is distributed uniformly along the projection of the cable on a horizontal straight line. The weight of the cable is neglected.

2964*. Find the position of equilibrium of a flexible nontensile thread, the ends of which are attached at two points and which has a constant load q (including the weight of the thread) per unit length.

2965*. A heavy body with no initial velocity is sliding along an inclined plane. Find the law of motion if the angle of inclination is α, and the coefficient of friction is μ.

(**Hint.** The frictional force is μN, where N is the force of reaction of the plane.)

2966*. We may consider that the air resistance in free fall is proportional to the square of the velocity. Find the law of motion if the initial velocity is zero.

2967*. A motor-boat weighing 300 kgf is in rectilinear motion with initial velocity 66 m/sec. The resistance of the water is proportional to the velocity and is 10 kgf at 1 metre/sec. How long will it be before the velocity becomes 8 m/sec?

Sec. 11. Linear Differential Equations

1°. **Homogeneous equations.** The functions $y_1=\varphi_1(x)$, $y_2=\varphi_2(x)$, ..., $y_n=\varphi_n(x)$ are called *linearly dependent* if there are constants $C_1, C_2, \ldots, C_n$ not all equal to zero, such that

$$C_1y_1+C_2y_2+\ldots+C_ny_n\equiv 0;$$

otherwise, these functions are called *linearly independent*.

The general solution of a *homogeneous linear* differential equation

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=0 \qquad (1)$$

with continuous coefficients $P_i(x)$ $(i=1, 2, \ldots, n)$ is of the form

$$y=C_1y_1+C_2y_2+\ldots+C_ny_n,$$

where $y_1, y_2, \ldots, y_n$ are linearly independent solutions of equation (1) (*fundamental system of solutions*).

2°. **Inhomogeneous equations.** The general solution of an *inhomogeneous linear* differential equation

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=f(x) \qquad (2)$$

with continuous coefficients $P_i(x)$ and the right side $f(x)$ has the form

$$y=y_0+Y,$$

where y_0 is the general solution of the corresponding homogeneous equation (1) and Y is a particular solution of the given inhomogeneous equation (2).

If the fundamental system of solutions $y_1, y_2, \ldots, y_n$ of the homogeneous equation (1) is known, then the general solution of the corresponding inhomogeneous equation (2) may be found from the formula

$$y=C_1(x)\,y_1+C_2(x)\,y_2+\ldots+C_n(x)\,y_n,$$

where the functions $C_i(x)$ $(i=1, 2, \ldots, n)$ are determined from the following system of equations:

$$\left.\begin{array}{l} C_1'(x)\,y_1+C_2'(x)\,y_2+\ldots+C_n'(x)\,y_1=0, \\ C_1'(x)\,y_1'+C_2'(x)\,y_2'+\ldots+C_n'(x)\,y_n'=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_1'(x)\,y_1^{(n-2)}+C_2'(x)\,y_2^{(n-2)}+\ldots+C_n'(x)\,y_n^{(n-2)}=0, \\ C_1'(x)\,y_1^{(n-1)}+C_2'(x)\,y_2^{(n-1)}+\ldots+C_n'(x)\,y_n^{(n-1)}=f(x) \end{array}\right\} \qquad (3)$$

(the *method of variation of parameters*).

Example. Solve the equation

$$xy''+y'=x^2. \qquad (4)$$

Solution. Solving the homogeneous equation

$$xy''+y'=0,$$

we get

$$y=C_1\ln x+C_2. \qquad (5)$$

Hence, it may be taken that

$$y_1=\ln x \text{ and } y_2=1$$

and the solution of equation (4) may be sought in the form

$$y=C_1(x)\ln x+C_2(x).$$

Forming the system (3) and taking into account that the reduced form of the equation (4) is $y''+\frac{y'}{x}=x$, we obtain

$$\begin{cases} C_1'(x)\ln x+C_2'(x)\,1=0, \\ C_1'(x)\,\frac{1}{x}+C_2'(x)\,0=x. \end{cases}$$

Whence

$$C_1(x)=\frac{x^3}{3}+A \quad \text{and} \quad C_2(x)=-\frac{x^3}{3}\ln x+\frac{x^3}{9}+B$$

and, consequently,

$$y=\frac{x^3}{9}+A\ln x+B,$$

where A and B are arbitrary constants.

2968. Test the following systems of functions for linear relationships:

a) $x,\ x+1$;
b) $x^2,\ -2x^2$;
c) $0,\ 1,\ x$;
d) $x,\ x+1,\ x+2$;
e) $x,\ x^2,\ x^3$;
f) $e^x,\ e^{2x},\ e^{3x}$;
g) $\sin x,\ \cos x,\ 1$;
h) $\sin^2 x,\ \cos^2 x,\ 1$.

2969. Form a linear homogeneous differential equation, knowing its fundamental system of equations:

a) $y_1=\sin x,\ y_2=\cos x$;
b) $y_1=e^x,\ y_2=xe^x$;
c) $y_1=x,\ y_2=x^2$.
d) $y_1=e^x,\ y_2=e^x\sin x,\ y_3=e^x\cos x$.

2970. Knowing the fundamental system of solutions of a linear homogeneous differential equation

$$y_1=x,\ y_2=x^2,\ y_3=x^3,$$

find its particular solution y that satisfies the initial conditions

$$y|_{x=1}=0,\quad y'|_{x=1}=-1,\quad y''|_{x=1}=2.$$

2971*. Solve the equation

$$y'' + \frac{2}{x} y' + y = 0,$$

knowing its particular solution $y_1 = \frac{\sin x}{x}$.

2972. Solve the equation

$$x^2 (\ln x - 1) y'' - xy' + y = 0,$$

knowing its particular solution $y_1 = x$.

By the method of variation of parameters, solve the following inhomogeneous linear equations.

2973. $x^2 y'' - xy' = 3x^3$.

2974*. $x^2 y'' + xy' - y = x^2$.

2975. $y''' + y' = \sec x$.

Sec. 12. Linear Differential Equations of Second Order with Constant Coefficients

1°. Homogeneous equations. A second-order linear equation with constant coefficients p and q without the right side is of the form

$$y'' + py' + qy = 0 \tag{1}$$

If k_1 and k_2 are roots of the characteristic equation

$$\varphi(k) \equiv k^2 + pk + q = 0, \tag{2}$$

then the general solution of equation (1) is written in one of the following three ways:

1) $y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$ if k_1 and k_2 are real and $k_1 \neq k_2$;
2) $y = e^{k_1 x} (C_1 + C_2 x)$ if $k_1 = k_2$;
3) $y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$ if $k_1 = \alpha + \beta i$ and $k_2 = \alpha - \beta i$ $(\beta \neq 0)$.

2°. Inhomogeneous equations. The general solution of a linear inhomogeneous differential equation

$$y'' + py' + qy = f(x) \tag{3}$$

may be written in the form of a sum:

$$y = y_0 + Y,$$

where y_0 is the general solution of the corresponding equation (1) without right side and determined from formulas (1) to (3), and Y is a particular solution of the given equation (3).

The function Y may be found by the *method of undetermined coefficients* in the following simple cases:

1. $f(x) = e^{ax} P_n(x)$, where $P_n(x)$ is a polynomial of degree n.

If a is not a root of the characteristic equation (2), that is, $\varphi(a) \neq 0$, then we put $Y = e^{ax} Q_n(x)$ where $Q_n(x)$ is a polynomial of degree n with undetermined coefficients.

If a is a root of the characteristic equation (2), that is, $\varphi(a) = 0$, then $Y = x^r e^{ax} Q_n(x)$, where r is the multiplicity of the root a ($r = 1$ or $r = 2$).

2. $f(x) = e^{ax} [P_n(x) \cos bx + Q_m(x) \sin bx]$.

If $\varphi(a \pm bi) \neq 0$, then we put

$$Y = e^{ax}[S_N(x)\cos bx + T_N(x)\sin bx],$$

where $S_N(x)$ and $T_N(x)$ are polynomials of degree N-max $\{n, m\}$.

But if $\varphi(a \pm bi) = 0$, then

$$Y = x^r e^{ax}[S_N(x)\cos bx + T_N(x)\sin bx],$$

where r is the multiplicity of the roots $a \pm bi$ (for second-order equations, $r = 1$).

In the general case, the *method of variation of parameters* (see Sec. 11) is used to solve equation (3).

Example 1. Find the general solution of the equation $2y'' - y' - y = 4xe^{2x}$.

Solution. The characteristic equation $2k^2 - k - 1 = 0$ has roots $k_1 = 1$ and $k_2 = -\frac{1}{2}$. The general solution of the corresponding homogeneous equation (first type) is $y_0 = C_1e^x + C_2e^{-\frac{x}{2}}$. The right side of the given equation is $f(x) = 4xe^{2x} = e^{ax}P_n(x)$. Hence, $Y = e^{2x}(Ax + B)$, since $n = 1$ and $r = 0$. Differentiating Y twice and putting the derivatives into the given equation, we obtain:

$$2e^{2x}(4Ax + 4B + 4A) - e^{2x}(2Ax + 2B + A) - e^{2x}(Ax + B) = 4xe^{2x}.$$

Cancelling out e^{2x} and equating the coefficients of identical powers of x and the absolute terms on the left and right of the equality, we have $5A = 4$ and $7A + 5B = 0$, whence $A = \frac{4}{5}$ and $B = -\frac{28}{25}$.

Thus, $Ye^{2x}\left(\frac{4}{5}x - \frac{28}{25}\right)$, and the general solution of the given equation is

$$y = C_1e^x + C_2e^{-\frac{1}{2}} + e^{2x}\left(\frac{4}{5}x - \frac{28}{25}\right).$$

Example 2. Find the general solution of the equation $y'' - 2y' + y = xe^x$.

Solution. The characteristic equation $k^2 - 2k + 1 = 0$ has a double root $k = 1$. The right side of the equation is of the form $f(x) = xe^x$; here, $a = 1$ and $n = 1$. The particular solution is $Y = x^2e^x(Ax + B)$, since a coincides with the double root $k = 1$ and, consequently, $r = 2$.

Differentiating Y twice, substituting into the equation, and equating the coefficients, we obtain $A = \frac{1}{6}$, $B = 0$. Hence, the general solution of the given equation will be written in the form

$$y = (C_1 + C_2x)e^x + \frac{1}{6}x^3e^x.$$

Example 3. Find the general solution of the equation $y'' + y = x \sin x$.

Solution. The characteristic equation $k^2 + 1 = 0$ has roots $k_1 = i$ and $k_2 = -i$. The general solution of the corresponding homogeneous equation will [see 3, where $\alpha = 0$ and $\beta = 1$] be

$$y_0 = C_1 \cos x + C_2 \sin x.$$

The right side is of the form

$$f(x) = e^{ax}[P_n(x)\cos bx + Q_m(x)\sin bx],$$

where $a=0$, $b=1$, $P_n(x)=0$, $Q_m(x)=x$. To this side there corresponds the particular solution Y,

$$Y=x[(Ax+B)\cos x+(Cx+D)\sin x]$$

(here, $N=1$, $a=0$, $b=1$, $r=1$).

Differentiating twice and substituting into the equation, we equate the coefficients of both sides in $\cos x$, $x\cos x$, $\sin x$, and $x\sin x$. We then get four equations $2A+2D=0$, $4C=0$, $-2B+2C=0$, $-4A=1$, from which we determine $A=-\frac{1}{4}$, $B=0$, $C=0$, $D=\frac{1}{4}$. Therefore, $Y=-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x$.

The general solution is

$$y=C_1\cos x+C_2\sin x-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x.$$

3°. The principle of superposition of solutions. If the right side of equation (3) is the sum of several functions

$$f(x)=f_1(x)+f_2(x)+\ldots+f_n(x)$$

and Y_i ($i=1, 2, 3, \ldots, n$) are the corresponding solutions of the equations

$$y''+py'+qy=f_i(x) \qquad (i=1, 2, \ldots, n),$$

then the sum

$$y=Y_1+Y_2+\ldots+Y_n$$

is the solution of equation (3).

Find the general solutions of the equations:

2976. $y''-5y'+6y=0$.

2977. $y''-9y=0$.

2978. $y''-y'=0$.

2979. $y''+y=0$.

2980. $y''-2y'+2y=0$.

2981. $y''+4y'+13y=0$.

2982. $y''+2y'+y=0$.

2983. $y''-4y'+2y=0$.

2984. $y''+ky=0$.

2985. $y=y''+y'$.

2986. $\frac{y'-y}{y''}=3$.

Find the particular solutions that satisfy the indicated conditions:

2987. $y''-5y'+4y=0$; $y=5$, $y'=8$ for $x=0$

2988. $y''+3y'+2y=0$; $y=1$, $y'=-1$ for $x=0$.

2989. $y''+4y=0$; $y=0$, $y'=2$ for $x=0$.

2990. $y''+2y'=0$; $y=1$, $y'=0$ for $x=0$

2991. $y''=\frac{y}{a^2}$; $y=a$, $y'=0$ for $x=0$.

2992. $y''+3y'=0$; $y=0$ for $x=0$ and $y=0$ for $x=3$.

2993. $y''+\pi^2y=0$; $y=0$ for $x=0$ and $y=0$ for $x=1$.

2994. Indicate the type of particular solutions for the given inhomogeneous equations:

a) $y''-4y=x^2e^{2x}$;

b) $y''+9y=\cos 2x$;

c) $y''-4y'+4y=\sin 2x+e^{2x}$;
d) $y''+2y'+2y=e^{x}\sin x$;
e) $y''-5y'+6y=(x^2+1)e^{x}+xe^{2x}$;
f) $y''-2y'+5y=xe^{x}\cos 2x-x^2e^{x}\sin 2x$.

Find the general solutions of the equations:

2995. $y''-4y'+4y=x^2$.
2996. $y''-y'+y=x^3+6$.
2997. $y''+2y'+y=e^{2x}$.
2998. $y''-8y'+7y=14$.
2999. $y''-y=e^{x}$.
3000. $y''+y=\cos x$.
3001. $y''+y'-2y=8\sin 2x$.
3002. $y''+y'-6y=xe^{2x}$.
3003. $y''-2y'+y=\sin x+\sinh x$.
3004. $y''+y'=\sin^2 x$.
3005. $y''-2y'+5y=e^{x}\cos 2x$.

3006. Find the solution of the equation $y''+4y=\sin x$ that satisfies the conditions $y=1$, $y'=1$ for $x=0$.

Solve the equations:

3007. $\frac{d^2x}{dt^2}+\omega^2x=A\sin pt$. Consider the cases: 1) $p\neq\omega$; 2) $p=\omega$.
3008. $y''-7y'+12y=-e^{4x}$.
3009. $y''-2y'=x^2-1$.
3010. $y''-2y'+y=2e^{x}$.
3011. $y''-2y'=e^{2x}+5$.
3012. $y''-2y'-8y=e^{x}-8\cos 2x$.
3013. $y''+y'=5x+2e^{x}$.
3014. $y''-y'=2x-1-3e^{x}$.
3015. $y''+2y'+y=e^{x}+e^{-x}$.
3016. $y''-2y'+10y=\sin 3x+e^{x}$.
3017. $y''-4y'+4y=2e^{2x}+\frac{x}{2}$.
3018. $y''-3y'=x+\cos x$.

3019. Find the solution to the equation $y''-2y'=e^{2x}+x^2-1$ that satisfies the conditions $y=\frac{1}{8}$, $y'=1$ for $x=0$.

Solve the equations:

3020. $y''-y=2x\sin x$.
3021. $y''-4y=e^{2x}\sin 2x$.
3022. $y''+4y=2\sin 2x-3\cos 2x+1$.
3023. $y''-2y'+2y=4e^{x}\sin x$.
3024. $y''=xe^{x}+y$.
3025. $y''+9y=2x\sin x+xe^{3x}$.

3026. $y''-2y'-3y=x(1+e^{3x})$.
3027. $y''-2y'=3x+2xe^{x}$.
3028. $y''-4y'+4y=xe^{2x}$.
3029. $y''+2y'-3y=2xe^{-3x}+(x+1)e^{x}$.
3030*. $y''+y=2x\cos x\cos 2x$.
3031. $y''-2y=2xe^{x}(\cos x-\sin x)$.

Applying the method of variation of parameters, solve the following equations:

3032. $y''+y=\tan x$.
3033. $y''+y=\cot x$.
3034. $y''-2y'+y=\frac{e^{x}}{x}$.
3035. $y''+2y'+y=\frac{e^{-x}}{x}$.
3036. $y''+y=\frac{1}{\cos x}$.
3037. $y''+y=\frac{1}{\sin x}$.
3038. a) $y''-y=\tanh x$.
b) $y''-2y=4x^{2}e^{x^{2}}$.

3039. Two identical loads are suspended from the end of a spring. Find the equation of motion that will be performed by one of these loads if the other falls.

Solution. Let the increase in the length of the spring under the action of one load in a state of rest be a and the mass of the load, m. Denote by x the coordinate of the load reckoned vertically from the position of equilibrium in the case of a single load. Then

$$m\frac{d^{2}x}{dt^{2}}=mg-k(x+a),$$

where, obviously, $k=\frac{mg}{a}$ and, consequently, $\frac{d^{2}x}{dt^{2}}=-\frac{g}{a}x$. The general solution is $x=C_1\cos\sqrt{\frac{g}{a}}\,t+C_2\sin\sqrt{\frac{g}{a}}\,t$. The initial conditions yield $x=a$ and $\frac{dx}{dt}=0$ when $t=0$; whence $C_1=a$ and $C_2=0$; and so

$$x=a\cos\sqrt{\frac{g}{a}}\,t.$$

3040*. The force stretching a spring is proportional to the increase in its length and is equal to 1 kgf when the length increases by 1 cm. A load weighing 2 kgf is suspended from the spring. Find the period of oscillatory motion of the load if it is pulled downwards slightly and then released.

3041*. A load weighing $P=4$ kgf is suspended from a spring and increases the length of the spring by 1 cm. Find the law of motion of the load if the upper end of the spring performs a vertical harmonic oscillation $y=2\sin 30t$ cm and if at the initial instant the load was at rest (resistance of the medium is neglected).

3042. A material point of mass m is attracted by each of two centres with a force proportional to the distance (the constant of proportionality is k). Find the law of motion of the point knowing that the distance between the centres is $2b$, at the initial instant the point was located on the line connecting the centres (at a distance c from its midpoint) and had a velocity of zero.

3043. A chain of length 6 metres is sliding from a support without friction. If the motion begins when 1 m of the chain is hanging from the support, how long will it take for the entire chain to slide down?

3044*. A long narrow tube is revolving with constant angular velocity ω about a vertical axis perpendicular to it. A ball inside the tube is sliding along it without friction. Find the law of motion of the ball relative to the tube, considering that

a) at the initial instant the ball was at a distance a from the axis of rotation; the initial velocity of the ball was zero;

b) at the initial instant the ball was located on the axis of rotation and had an initial velocity v_0.

Sec. 13. Linear Differential Equations of Order Higher than Two with Constant Coefficients

1°. Homogeneous equations. The fundamental system of solutions $y_1, y_2, \ldots, y_n$ of a homogeneous linear equation with constant coefficients

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny=0 \tag{1}$$

is constructed on the basis of the character of the roots of the *characteristic equation*

$$k^n+a_1k^{n-1}+\ldots+a_{n-1}k+a_n=0. \tag{2}$$

Namely, 1) if k is a real root of the equation (2) of multiplicity m, then to this root there correspond m linearly independent solutions of equation (1):

$$y_1=e^{kx},\ y_2=xe^{kx},\ \ldots,\ y_m=x^{m-1}e^{kx};$$

2) if $\alpha \pm \beta i$ is a pair of complex roots of equation (2) of multiplicity m, then to the latter there correspond $2m$ linearly independent solutions of equation (1):

$$y_1=e^{\alpha x}\cos\beta x,\ y_2=e^{\alpha x}\sin\beta x,\ y_3=xe^{\alpha x}\cos\beta x,\ y_4=xe^{\alpha x}\sin\beta x,\ \ldots$$
$$\ldots,\ y_{2m-1}=x^{m-1}e^{\alpha x}\cos\beta x,\ y_{2m}=x^{m-1}e^{\alpha x}\sin\beta x.$$

2°. Inhomogeneous equations. A particular solution of the inhomogeneous equation

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny=f(x) \tag{3}$$

is sought on the basis of rules 2° and 3° of Sec. 12.

Find the general solutions of the equations:

3045. $y''' - 13y'' + 12y' = 0$.

3046. $y''' - y' = 0$.

3047. $y''' + y = 0$.

3048. $y^{IV} - 2y'' = 0$.

3049. $y''' - 3y'' + 3y' - y = 0$.

3050. $y^{IV} + 4y = 0$.

3051. $y^{IV} + 8y'' + 16y = 0$.

3052. $y^{IV} + y' = 0$.

3053. $y^{IV} - 2y'' + y = 0$.

3054. $y^{IV} - a^4 y = 0$.

3055. $y^{IV} - 6y'' + 9y = 0$.

3056. $y^{IV} + a^2 y'' = 0$.

3057. $y^{IV} + 2y''' + y'' = 0$.

3058. $y^{IV} + 2y'' + y = 0$.

3059. $y^{(n)} + \frac{n}{1} y^{(n-1)} + \frac{n(n-1)}{1\cdot 2} y^{(n-2)} + \ldots + \frac{n}{1} y' + y = 0$.

3060. $y^{IV} - 2y''' + y'' = e^x$.

3061. $y^{IV} - 2y''' + y'' = x^3$.

3062. $y''' - y = x^3 - 1$.

3063. $y^{IV} + y''' = \cos 4x$.

3064. $y''' + y'' = x^2 + 1 + 3xe^x$.

3065. $y''' + y'' + y' + y = xe^x$.

3066. $y''' + y' = \tan x \sec x$.

3067. Find the particular solution of the equation

$$y''' + 2y'' + 2y' + y = x$$

that satisfies the initial conditions $y(0) = y'(0) = y''(0) = 0$.

Sec. 14. Euler's Equations

A linear equation of the form

$$(ax+b)^n y^{(n)} + A_1 (ax+b)^{n-1} y^{(n-1)} + \ldots + A_{n-1}(ax+b) y + A_n y = f(x), \quad (1)$$

where a, b, A_1, ..., A_{n-1}, A_n are constants, is called *Euler's equation*.

Let us introduce a new independent variable t, putting

$$ax + b = e^t.$$

Then

$$y' = ae^{-t}\frac{dy}{dt}, \quad y'' = a^2 e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right),$$

$$y''' = a^3 e^{-3t}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) \text{ and so forth}$$

and Euler's equation is transformed into a linear equation with constant coefficients.

Example 1. Solve the equation $x^2y'' + xy' + y = 1$.

Solution. Putting $x = e^t$, we get

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Consequently, the given equation takes on the form

$$\frac{d^2y}{dt^2} + y = 1,$$

whence

$$y = C_1 \cos t + C_2 \sin t + 1$$

or

$$y = C_1 \cos(\ln x) + C_2 \sin(\ln x) + 1.$$

For the homogeneous Euler equation

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \ldots + A_{n-1} x y' + A_n y = 0 \tag{2}$$

the solution may be sought in the form

$$y = x^k. \tag{3}$$

Putting into (2) y, y', ..., $y^{(n)}$ found from (3), we get a characteristic equation from which we can find the exponent k.

If k is a real root of the characteristic equation of multiplicity m, then to it correspond m linearly independent solutions

$$y_1 = x^k,\ y_2 = x^k \cdot \ln x,\ y_3 = x^k (\ln x)^2,\ \ldots,\ y_m = x^k (\ln x)^{m-1}.$$

If $\alpha \pm \beta i$ is a pair of complex roots of multiplicity m, then to it there correspond $2m$ linearly independent solutions

$$y_1 = x^\alpha \cos(\beta \ln x),\ y_2 = x^\alpha \sin(\beta \ln x),\ y_3 = x^\alpha \ln x \cos(\beta \ln x),$$
$$y_4 = x^\alpha \ln x \cdot \sin(\beta \ln x),\ \ldots,\ y_{2m-1} = x^\alpha (\ln x)^{m-1} \cos(\beta \ln x),$$
$$y_{2m} = x^\alpha (\ln x)^{m-1} \sin(\beta \ln x).$$

Example 2. Solve the equation

$$x^2 y'' - 3xy' + 4y = 0.$$

Solution. We put

$$y = x^k,\quad y' = kx^{k-1},\quad y'' = k(k-1)x^{k-2}.$$

Substituting into the given equation, after cancelling out x^k, we get the characteristic equation

$$k^2 - 4k + 4 = 0.$$

Solving it we find

$$k_1 = k_2 = 2.$$

Hence, the general solution will be

$$y = C_1 x^2 + C_2 x^2 \ln x.$$

Solve the equations:

3068. $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = 0.$

3069. $x^2 y'' - xy' - 3y = 0.$

3070. $x^2 y'' + xy' + 4y = 0.$

3071. $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0.$

3072. $(3x+2)y'' + 7y' = 0.$

3073. $y'' = \dfrac{2y}{x^2}.$

3074. $y'' + \dfrac{y'}{x} + \dfrac{y}{x^2} = 0.$

3075. $x^2 y'' - 4xy' + 6y = x.$

3076. $(1+x)^2 y'' - 3(1+x)y' + 4y = (1+x)^3.$

3077. Find the particular solution of the equation

$$x^2y''-xy'+y=2x$$

that satisfies the initial conditions $y=0$, $y'=1$ when $x=1$.

Sec. 15. Systems of Differential Equations

Method of elimination. To find the solution, for instance, of a normal system of two first-order differential equations, that is, of a system of the form

$$\frac{dy}{dx}=f(x, y, z), \qquad \frac{dz}{dx}=g(x, y, z), \tag{1}$$

solved for the derivatives of the desired functions, we differentiate one of them with respect to x. We have, for example,

$$\frac{d^2y}{dx^2}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f+\frac{\partial f}{\partial z}g. \tag{2}$$

Determining z from the first equation of the system (1) and substituting the value found,

$$z=\varphi\left(x, y, \frac{dy}{dx}\right) \tag{3}$$

into equation (2), we get a second-order equation with one unknown function y. Solving it, we find

$$y=\psi(x, C_1, C_2), \tag{4}$$

where C_1 and C_2 are arbitrary constants. Substituting function (4) into formula (3), we determine the function z without new integrations. The set of formulas (3) and (4), where y is replaced by ψ, yields the *general solution of the system* (1).

Example. Solve the system

$$\begin{cases} \dfrac{dy}{dx}+2y+4z=1+4x, \\ \dfrac{dz}{dx}+y-z=\dfrac{3}{2}x^2. \end{cases}$$

Solution. We differentiate the first equation with respect to x:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4\frac{dz}{dx}=4.$$

From the first equation we determine $z=\frac{1}{4}\left(1+4x-\frac{dy}{dx}-2y\right)$ and then from the second we will have $\frac{dz}{dx}=\frac{3}{2}x^2+x+\frac{1}{4}-\frac{3}{2}y-\frac{1}{4}\frac{dy}{dx}$. Putting z and $\frac{dz}{dx}$ into the equation obtained after differentiation, we arrive at a second-order equation in one unknown y:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=-6x^2-4x+3.$$

Solving it we find:

$$y=C_1e^{2x}+C_2e^{-3x}+x^2+x,$$

and then

$$z=\frac{1}{4}\left(1+4x-\frac{dy}{dx}-2y\right)=-C_1e^{2x}+\frac{C_2}{4}e^{-3x}-\frac{1}{2}x^2.$$

We can do likewise in the case of a system with a larger number of equations.

Solve the systems:

3078. $\begin{cases} \dfrac{dy}{dx}=z, \\ \dfrac{dz}{dx}=-y. \end{cases}$

3079. $\begin{cases} \dfrac{dy}{dx}=y+5z, \\ \dfrac{dz}{dx}+y+3z=0. \end{cases}$

3080. $\begin{cases} \dfrac{dy}{dx}=-3y-z, \\ \dfrac{dz}{dx}=y-z. \end{cases}$

3081. $\begin{cases} \dfrac{dx}{dt}=y, \\ \dfrac{dy}{dt}=z, \\ \dfrac{dz}{dt}=x. \end{cases}$

3082. $\begin{cases} \dfrac{dx}{dt}=y+z, \\ \dfrac{dy}{dt}=x+z, \\ \dfrac{dz}{dt}=x+y. \end{cases}$

3083. $\begin{cases} \dfrac{dy}{dx}=y+z, \\ \dfrac{dz}{dx}=x+y+z. \end{cases}$

3084. $\begin{cases} \dfrac{dy}{dx}+2y+z=\sin x, \\ \dfrac{dz}{dx}-4y-2z=\cos x. \end{cases}$

3085. $\begin{cases} \dfrac{dy}{dx}+3y+4z=2x, \\ \dfrac{dz}{dx}-y-z=x, \end{cases}$

$y=0$, $z=0$ when $x=0$.

3086. $\begin{cases} \dfrac{dx}{dt}-4x-y+36t=0, \\ \dfrac{dy}{dt}+2x-y+2e^t=0, \end{cases}$

$x=0$, $y=1$ when $t=0$.

3087. $\begin{cases} \dfrac{dy}{dx}=\dfrac{y^2}{z}, \\ \dfrac{dz}{dx}=\dfrac{1}{2}y. \end{cases}$

3088*. a) $\dfrac{dx}{x^3+3xy^2}=\dfrac{dy}{2y^3}=\dfrac{dz}{2y^2z}$;

b) $\dfrac{dx}{x-y}=\dfrac{dy}{x+y}=\dfrac{dz}{z}$;

c) $\dfrac{dx}{y-z}=\dfrac{dy}{z-x}=\dfrac{dz}{x-y}$,

isolate the integral curve passing through the point $(1, 1, -2)$.

3089. $\begin{cases} \dfrac{dy}{dx}+z=1, \\ \dfrac{dz}{dx}+\dfrac{2}{x^2}y=\ln x. \end{cases}$

3090. $\begin{cases} \dfrac{d^2y}{dx^2}+2y+4z=e^x, \\ \dfrac{d^2z}{dx^2}-y-3z=-x. \end{cases}$

3091.** A shell leaves a gun with initial velocity v_0 at an angle α to the horizon. Find the equation of motion if we take the air resistance as proportional to the velocity.

3092*. A material point is attracted by a centre O with a force proportional to the distance. The motion begins from point A at a distance a from the centre with initial velocity v_0 perpendicular to OA. Find the trajectory.

Sec. 16. Integration of Differential Equations by Means of Power Series

If it is not possible to integrate a differential equation with the help of elementary functions, then in some cases its solution may be sought in the form of a power series:

$$y=\sum_{n=0}^{\infty} c_n (x-x_0)^n. \tag{1}$$

The undetermined coefficients c_n $(n=1, 2, \ldots)$ are found by putting the series (1) into the equation and equating the coefficients of identical powers of the binomial $x-x_0$ on the left-hand and right-hand sides of the resulting equation.

We can also seek the solution of the equation

$$y'=f(x, y); \quad y(x_0)=y_0 \tag{2}$$

in the form of the Taylor's series

$$y(x)=\sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n, \tag{3}$$

where $y(x_0)=y_0$, $y'(x_0)=f(x_0, y_0)$ and the subsequent derivatives $y^{(n)}(x_0)$ $(n=2, 3, \ldots)$ are successively found by differentiating equation (2) and by putting x_0 in place of x

Example 1. Find the solution of the equation

$$y''-xy=0,$$

if $y=y_0$, $y'=y_0'$ for $x=0$.

Solution. We put

$$y=c_0+c_1x+\ldots+c_nx^n+\ldots,$$

whence, differentiating, we get

$$y''=2\cdot 1c_2+3\cdot 2c_3x+\ldots+n(n-1)c_nx^{n-2}+(n+1)nc_{n+1}x^{n-1}+ \\ +(n+2)(n+1)c_{n+2}x^n+\ldots$$

Substituting y and y'' into the given equation, we arrive at the identity

$$[2\cdot 1c_2+3\cdot 2c_3x+\ldots+n(n-1)c_nx^{n-2}+(n+1)nc_{n+1}x^{n-1}+ \\ +(n+2)(n+1)c_{n+2}x^n+\ldots]-x[c_0+c_1x+\ldots+c_nx^n+\ldots]\equiv 0.$$

Collecting together, on the left of this equation, the terms with identical powers of x and equating to zero the coefficients of these powers, we will have

$$c_2=0;\quad 3\cdot 2c_3-c_0=0,\quad c_3=\frac{c_0}{3\cdot 2};\quad 4\cdot 3c_4-c_1=0,\ c_4=\frac{c_1}{4\cdot 3};\quad 5\cdot 4c_5-c_2=0,$$

$$c_5=\frac{c_2}{5\cdot 4}\ \text{and so forth.}$$

Generally,

$$c_{3k}=\frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3k-1)\,3k},\qquad c_{3k+1}=\frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot\ldots\cdot 3k\,(3k+1)},$$

$$c_{3k+2}=0\quad (k=1,\ 2,\ 3,\ \ldots).$$

Consequently,

$$y=c_0\left(1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\ldots+\frac{x^{3k}}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3k-1)\,3k}+\ldots\right)+$$
$$+c_1\left(x+\frac{x^4}{3\cdot 4}+\frac{x^7}{3\cdot 4\cdot 6\cdot 7}+\ldots+\frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdot\ldots\cdot 3k\,(3k+1)}+\ldots\right), \quad (4)$$

where $c_0=y_0$ and $c_1=y_0'$.

Applying d'Alembert's test, it is readily seen that series (4) converges for $-\infty<x<+\infty$.

Example 2. Find the solution of the equation

$$y'=x+y;\quad y_0=y\,(0)=1.$$

Solution. We put

$$y=y_0+y_0'x+\frac{y_0''}{2!}x^2+\frac{y_0'''}{3!}x^3+\ldots$$

We have $y_0=1$, $y_0'=0+1=1$. Differentiating equation $y'=x+y$, we successively find $y''=1+y'$, $y_0''=1+1=2$, $y'''=y''$, $y_0'''=2$, etc. Consequently,

$$y=1+x+\frac{2}{2!}x^2+\frac{2}{3!}x^3+\ldots$$

For the example at hand, this solution may be written in final form as

$$y=1+x+2\,(e^x-1-x)\ \text{or}\ y=2e^x-1-x.$$

The procedure is similar for differential equations of higher orders. Testing the resulting series for convergence is, generally speaking, complicated and is not obligatory when solving the problems of this section.

With the help of power series, find the solutions of the equations for the indicated initial conditions.

In Examples 3097, 3098, 3099, 3101, test the solutions obtained for convergence.

3093. $y'=y+x^2$; $y=-2$ for $x=0$.

3094. $y'=2y+x-1$; $y=y_0$ for $x=1$.

3095. $y'=y^2+x^3$; $y=\frac{1}{2}$ for $x=0$.

3096. $y'=x^2-y^2$; $y=0$ for $x=0$.

3097. $(1-x)\,y'=1+x-y$; $y=0$ for $x=0$.

3098*. $xy'' + y = 0$; $y = 0$, $y' = 1$ for $x = 0$.
3099. $y'' + xy = 0$; $y = 1$, $y' = 0$ for $x = 0$.
3100*. $y'' + \frac{2}{x} y' + y = 0$; $y = 1$, $y' = 0$ for $x = 0$.
3101*. $y'' + \frac{1}{x} y' + y = 0$; $y = 1$, $y' = 0$ for $x = 0$.
3102. $\frac{d^2x}{dt^2} + x \cos t = 0$; $x = a$; $\frac{dx}{dt} = 0$ for $t = 0$.

Sec. 17. Problems on Fourier's Method

To find the solutions of a linear homogeneous partial differential equation by Fourier's method, first seek the particular solutions of this special-type equation, each of which represents the product of functions that are dependent on one argument only. In the simplest case, there is an infinite set of such solutions u_n ($n = 1, 2, \ldots$), which are linearly independent among themselves in any finite number and which satisfy the given *boundary conditions*. The desired solution u is represented in the form of a series arranged according to these particular solutions:

$$u = \sum_{n=1}^{\infty} C_n u_n. \tag{1}$$

The coefficients C_n which remain undetermined are found from the *initial conditions*.

Problem. A transversal displacement $u = u(x, t)$ of the points of a string with abscissa x satisfies, at time t, the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

where $a^2 = \frac{T_0}{\varrho}$ (T_0 is the tensile force and ϱ is the linear density of the string). Find the form of the string at time t if its ends $x = 0$ and $x = l$ are

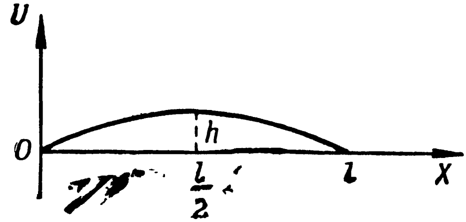

Fig. 107

fixed and at the initial instant, $t = 0$, the string had the form of a parabola $u = \frac{4h}{l^2} x(l - x)$ (Fig. 107) and its points had zero velocity.

Solution. It is required to find the solution $u = u(x, t)$ of equation (2) that satisfies the boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0 \tag{3}$$

and the initial conditions

$$u(x, 0)=\frac{4h}{l^2}x(l-x), \quad u_t'(x, 0)=0. \tag{4}$$

We seek the nonzero solutions of equation (2) of the special form

$$u=X(x)\,T(t).$$

Putting this expression into equation (2) and separating the variables, we get

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}. \tag{5}$$

Since the variables x and t are independent, equation (5) is possible only when the general quantity of relation (5) is constant Denoting this constant by $-\lambda^2$, we find two ordinary differential equations:

$$T''(t)+(a\lambda)^2\cdot T(t)=0 \text{ and } X''(x)+\lambda^2X(x)=0.$$

Solving these equations, we get

$$T(t)=A\cos a\lambda t+B\sin a\lambda t,$$
$$X(x)=C\cos\lambda x+D\sin\lambda x,$$

where A, B, C, D are arbitrary constants. Let us determine the constants. From condition (3) we have $X(0)=0$ and $X(l)=0$; hence, $C=0$ and $\sin\lambda l=0$ (since D cannot be equal to zero at the same time as C is zero). For this reason, $\lambda_k=\frac{k\pi}{l}$, where k is an integer. It will readily be seen that we do not lose generality by taking for k only positive values ($k=1, 2, 3, \ldots$).

To every value λ_k there corresponds a particular solution

$$u_k=\left(A_k\cos\frac{ka\pi}{l}t+B_k\sin\frac{ka\pi}{l}t\right)\sin\frac{k\pi x}{l}$$

that satisfies the boundary conditions (3).

We construct the series

$$u=\sum_{k=1}^{\infty}\left(A_k\cos\frac{ka\pi t}{l}+B_k\sin\frac{ka\pi t}{l}\right)\sin\frac{k\pi x}{l},$$

whose sum obviously satisfies equation (2) and the boundary conditions (3).

We choose the constants A_k and B_k so that the sum of the series should satisfy the initial conditions (4). Since

$$\frac{\partial u}{\partial t}=\sum_{k=1}^{\infty}\frac{ka\pi}{l}\left(-A_k\sin\frac{ka\pi t}{l}+B_k\cos\frac{ka\pi t}{l}\right)\sin\frac{k\pi x}{l},$$

it follows that, by putting $t=0$, we obtain

$$u(x, 0)=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi x}{l}\equiv\frac{4h}{l^2}x(l-x)$$

and

$$\frac{\partial u(x, 0)}{\partial t}=\sum_{k=1}^{\infty}\frac{ka\pi}{l}B_k\sin\frac{k\pi x}{l}\equiv 0.$$

Hence, to determine the coefficients A_k and B_k it is necessary to expand in a Fourier series, in sines only, the function $u(x, 0)=\frac{4h}{l^2}x(l-x)$ and the function $\frac{\partial u(x, 0)}{\partial t}\equiv 0$.

From familiar formulas (Ch. VIII, Sec. 4,3°) we have

$$A_k=\frac{2}{l}\int_0^l \frac{4h}{l^2}x(l-x)\sin\frac{k\pi x}{l}\,dx=\frac{32h}{\pi^3 k^3},$$

if k is odd, and $A_k=0$ if k is even;

$$\frac{ka\pi}{l}B_k=\frac{2}{l}\int_0^l 0\sin\frac{k\pi x}{l}\,dx=0,\quad B_k=0.$$

The sought-for solution will be

$$u=\frac{32h}{\pi^3}\sum_{n=0}^{\infty}\frac{\cos\frac{(2n+1)a\pi t}{l}}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{l}.$$

3103*. At the initial instant $t=0$, a string, attached at its ends, $x=0$ and $x=l$, had the form of the sine curve $u=A\sin\frac{\pi x}{l}$, and the points of it had zero velocity. Find the form of the string at time t.

3104*. At the initial time $t=0$, the points of a straight string $0<x<l$ receive a velocity $\frac{\partial u}{\partial t}=1$. Find the form of the string at time t if the ends of the string $x=0$ and $x=l$ are fixed (see Problem 3103).

3105*. A string of length $l=100$ cm and attached at its ends, $x=0$ and $x=l$, is pulled out to a distance $h=2$ cm at point $x=50$ cm at the initial time, and is then released without any impulse. Determine the shape of the string at any time t.

3106*. In longitudinal vibrations of a thin homogeneous and rectilinear rod, whose axis coincides with the x-axis, the displacement $u=u(x, t)$ of a cross-section of the rod with abscissa x satisfies, at time t, the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},$$

where $a^2=\frac{E}{\varrho}$ (E is Young's modulus and ϱ is the density of the rod). Determine the longitudinal vibrations of an elastic horizontal rod of length $l=100$ cm fixed at the end $x=0$ and pulled back at the end $x=100$ by $\Delta l=1$ cm, and then released without impulse.

3107*. For a rectilinear homogeneous rod whose axis coincides with the x-axis, the temperature $u=u(x,\ t)$ in a cross-section with abscissa x at time t, in the absence of sources of heat, satisfies the equation of heat conduction

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},$$

where a is a constant. Determine the temperature distribution for any time t in a rod of length 100 cm if we know the initial temperature distribution

$$u(x,\ 0)=0.01\,x\,(100-x).$$