Chapter II

DIFFERENTIATION OF FUNCTIONS

Sec. 1. Calculating Derivatives Directly

1°. Increment of the argument and increment of the function. If x and x_1 are values of the argument x, and $y=f(x)$ and $y_1=f(x_1)$ are corresponding values of the function $y=f(x)$, then

$$\Delta x = x_1 - x$$

is called the *increment of the argument* x in the interval (x, x_1), and

$$\Delta y = y_1 - y$$

or

$$\Delta y = f(x_1) - f(x) = f(x+\Delta x) - f(x) \tag{1}$$

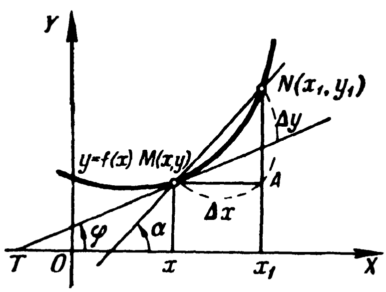

Fig. 11

is called the *increment of the function* y in the same interval (x, x_1) (Fig. 11, where $\Delta x = MA$ and $\Delta y = AN$). The ratio

$$\frac{\Delta y}{\Delta x} = \tan \alpha$$

is the slope of the secant MN of the graph of the function $y=f(x)$ (Fig. 11) and is called the *mean rate of change* of the function y over the interval $(x, x+\Delta x)$.

Example 1. For the function

$$y = x^2 - 5x + 6$$

calculate Δx and Δy, corresponding to a change in the argument:

a) from $x=1$ to $x=1.1$;
b) from $x=3$ to $x=2$.

Solution. We have

a) $\Delta x = 1.1-1=0.1$,
$\Delta y=(1.1^2-5\cdot 1.1+6)-(1^2-5\cdot 1+6)=-0.29$;
b) $\Delta x=2-3=-1$,
$\Delta y=(2^2-5\cdot 2+6)-(3^2-5\cdot 3+6)=0$.

Example 2. In the case of the hyperbola $y=\frac{1}{x}$, find the slope of the secant passing through the points $M\left(3, \frac{1}{3}\right)$ and $N\left(10, \frac{1}{10}\right)$.

Solution. Here, $\Delta x=10-3=7$ and $\Delta y=\frac{1}{10}-\frac{1}{3}=-\frac{7}{30}$. Hence, $k=\frac{\Delta y}{\Delta x}=-\frac{1}{30}$.

2°. The derivative. *The derivative* $y'=\frac{dy}{dx}$ of a function $y=f(x)$ with respect to the argument x is the limit of the ratio $\frac{\Delta y}{\Delta x}$ when Δx approaches zero; that is,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

The magnitude of the derivative yields the *slope* of the tangent MT to the graph of the function $y=f(x)$ at the point x (Fig. 11):

$$y'=\tan\varphi.$$

Finding the derivative y' is usually called *differentiation of the function*. The derivative $y'=f'(x)$ is the *rate of change of the function* at the point x.

Example 3. Find the derivative of the function

$$y=x^2.$$

Solution. From formula (1) we have

$$\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+(\Delta x)^2$$

and

$$\frac{\Delta y}{\Delta x}=2x+\Delta x.$$

Hence,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x.$$

3°. One-sided derivatives. The expressions

$$f'_-(x)=\lim_{\Delta x\to -0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

and

$$f'_+(x)=\lim_{\Delta x\to +0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

are called, respectively, the *left-hand* or *right-hand derivative* of the function $f(x)$ at the point x. For $f'(x)$ to exist, it is necessary and sufficient that

$$f'_-(x) = f'_+(x).$$

Example 4 Find $f'_-(0)$ and $f'_+(0)$ of the function

$$f(x) = |x|.$$

Solution. By the definition we have

$$f'_-(0) = \lim_{\Delta x \to -0} \frac{|\Delta x|}{\Delta x} = -1,$$

$$f'_+(0) = \lim_{\Delta x \to +0} \frac{|\Delta x|}{\Delta x} = 1.$$

4°. **Infinite derivative.** If at some point we have

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \infty,$$

then we say that the continuous function $f(x)$ has an infinite derivative at x. In this case, the tangent to the graph of the function $y = f(x)$ is perpendicular to the x-axis.

Example 5. Find $f'(0)$ of the function

$$y = \sqrt[3]{x}.$$

Solution. We have

$$f'(0) = \lim_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt[3]{\Delta x^2}} = \infty.$$

341. Find the increment of the function $y = x^2$ that corresponds to a change in argument:

a) from $x = 1$ to $x_1 = 2$;
b) from $x = 1$ to $x_1 = 1.1$;
c) from $x = 1$ to $x_1 = 1 + h$.

342. Find Δy of the function $y = \sqrt[3]{x}$ if:

a) $x = 0$, $\Delta x = 0.001$;
b) $x = 8$, $\Delta x = -9$;
c) $x = a$, $\Delta x = h$.

343. Why can we, for the function $y = 2x + 3$, determine the increment Δy if all we know is the corresponding increment $\Delta x = 5$, while for the function $y = x^2$ this cannot be done?

344. Find the increment Δy and the ratio $\frac{\Delta y}{\Delta x}$ for the functions:

a) $y = \frac{1}{(x^2-2)^2}$ for $x = 1$ and $\Delta x = 0.4$;
b) $y = \sqrt{x}$ for $x = 0$ and $\Delta x = 0.0001$;
c) $y = \log x$ for $x = 100,000$ and $\Delta x = -90,000$.

345. Find Δy and $\frac{\Delta y}{\Delta x}$ which correspond to a change in argument from x to $x+\Delta x$ for the functions:

a) $y=ax+b$; d) $y=\sqrt{x}$;
b) $y=x^3$; e) $y=2^x$;
c) $y=\frac{1}{x^2}$; f) $y=\ln x$.

346. Find the slope of the secant to the parabola

$$y=2x-x^2,$$

if the abscissas of the points of intersection are equal:

a) $x_1=1$, $x_2=2$;
b) $x_1=1$, $x_2=0.9$;
c) $x_1=1$, $x_2=1+h$.

To what limit does the slope of the secant tend in the latter case if $h \to 0$?

347. What is the mean rate of change of the function $y=x^3$ in the interval $1 \leqslant x \leqslant 4$?

348. The law of motion of a point is $s=2t^2+3t+5$, where the distance s is given in centimetres and the time t is in seconds. What is the average velocity of the point over the interval of time from $t=1$ to $t=5$?

349. Find the mean rise of the curve $y=2^x$ in the interval $1 \leqslant x \leqslant 5$.

350. Find the mean rise of the curve $y=f(x)$ in the interval $[x, x+\Delta x]$.

351. What is to be understood by the rise of the curve $y=f(x)$ at a given point x?

352. Define: a) the mean rate of rotation; b) the instantaneous rate of rotation.

353. A hot body placed in a medium of lower temperature cools off. What is to be understood by: a) the mean rate of cooling; b) the rate of cooling at a given instant?

354. What is to be understood by the rate of reaction of a substance in a chemical reaction?

355. Let $m=f(x)$ be the mass of a non-homogeneous rod over the interval $[0, x]$. What is to be understood by: a) the mean linear density of the rod on the interval $[x, x+\Delta x]$; b) the linear density of the rod at a point x?

356. Find the ratio $\frac{\Delta y}{\Delta x}$ of the function $y=\frac{1}{x}$ at the point $x=2$, if: a) $\Delta x=1$; b) $\Delta x=0.1$; c) $\Delta x=0.01$. What is the derivative y' when $x=2$?

357**. Find the derivative of the function $y=\tan x$.

358. Find $y'=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ of the functions:

a) $y=x^3$; c) $y=\sqrt{x}$;
b) $y=\frac{1}{x^2}$; d) $y=\cot x$.

359. Calculate $f'(8)$, if $f(x)=\sqrt[3]{x}$.

360. Find $f'(0)$, $f'(1)$, $f'(2)$, if $f(x)=x(x-1)^2(x-2)^3$.

361. At what points does the derivative of the function $f(x)=x^3$ coincide numerically with the value of the function itself, that is, $f(x)=f'(x)$?

362. The law of motion of a point is $s=5t^2$, where the distance s is in metres and the time t is in seconds. Find the speed at $t=3$.

363. Find the slope of the tangent to the curve $y=0.1x^3$ drawn at a point with abscissa $x=2$.

364. Find the slope of the tangent to the curve $y=\sin x$ at the point $(\pi, 0)$.

365. Find the value of the derivative of the function $f(x)=\frac{1}{x}$ at the point $x=x_0$ $(x_0\neq 0)$.

366*. What are the slopes of the tangents to the curves $y=\frac{1}{x}$ and $y=x^2$ at the point of their intersection? Find the angle between these tangents.

367**. Show that the following functions do not have finite derivatives at the indicated points:

a) $y=\sqrt[3]{x^2}$ at $x=0$;
b) $y=\sqrt[5]{x-1}$ at $x=1$;
c) $y=|\cos x|$ at $x=\frac{2k+1}{2}\pi$, $k=0, \pm 1, \pm 2, \ldots$

Sec. 2. Tabular Differentiation

1°. Basic rules for finding a derivative. If c is a constant and $u=\varphi(x)$, $v=\psi(x)$ are functions that have derivatives, then

1) $(c)'=0$;
2) $(x)'=1$;
3) $(u\pm v)'=u'\pm v'$;
4) $(cu)'=cu'$;
5) $(uv)'=u'v+v'u$;
6) $\left(\frac{u}{v}\right)'=\frac{vu'-v'u}{v^2}$ $(v\neq 0)$;
7) $\left(\frac{c}{v}\right)'=\frac{-cv'}{v^2}$ $(v\neq 0)$.

2°. Table of derivatives of basic functions

I. $(x^n)'=nx^{n-1}$.

II. $(\sqrt{x})'=\frac{1}{2\sqrt{x}}$ $(x>0)$.

III. $(\sin x)'=\cos x$.

IV. $(\cos x)'=-\sin x$.

V. $(\tan x)'=\frac{1}{\cos^2 x}$.

VI. $(\cot x)'=\frac{-1}{\sin^2 x}$.

VII. $(\operatorname{arc}\sin x)'=\frac{1}{\sqrt{1-x^2}}$ $(|x|<1)$.

VIII. $(\operatorname{arc}\cos x)'=\frac{-1}{\sqrt{1-x^2}}$ $(|x|<1)$.

IX. $(\operatorname{arc}\tan x)'=\frac{1}{1+x^2}$.

X. $(\operatorname{arc}\cot x)'=\frac{-1}{x^2+1}$.

XI. $(a^x)'=a^x\ln a$.

XII. $(e^x)'=e^x$.

XIII. $(\ln x)'=\frac{1}{x}$ $(x>0)$.

XIV. $(\log_a x)'=\frac{1}{x\ln a}=\frac{\log_a e}{x}$ $(x>0,\ a>0)$.

XV. $(\sinh x)'=\cosh x$.

XVI. $(\cosh x)'=\sinh x$.

XVII. $(\tanh x)'=\frac{1}{\cosh^2 x}$.

XVIII. $(\coth x)'=\frac{-1}{\sinh^2 x}$.

XIX. $(\operatorname{arc}\sinh x)'=\frac{1}{\sqrt{1+x^2}}$.

XX. $(\operatorname{arc}\cosh x)'=\frac{1}{\sqrt{x^2-1}}$ $(|x|>1)$.

XXI. $(\operatorname{arc}\tanh x)'=\frac{1}{1-x^2}$ $(|x|<1)$.

XXII. $(\operatorname{arc}\coth x)'=\frac{-1}{x^2-1}$ $(|x|>1)$.

3°. Rule for differentiating a composite function. If $y=f(u)$ and $u=\varphi(x)$, that is, $y=f[\varphi(x)]$, where the functions y and u have derivatives, then

$$y'_x=y'_u u'_x \tag{1}$$

or in other notations

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}.$$

This rule extends to a series of any finite number of differentiable functions.

Example 1. Find the derivative of the function

$$y=(x^2-2x+3)^5.$$

Solution. Putting $y=u^5$, where $u=(x^2-2x+3)$, by formula (1) we will have

$$y'=(u^5)'_u (x^2-2x+3)'_x=5u^4 (2x-2)=10 (x-1) (x^2-2x+3)^4.$$

Example 2. Find the derivative of the function

$$y=\sin^3 4x.$$

Solution. Putting

$$y=u^3; \quad u=\sin v; \quad v=4x,$$

we find

$$y'=3u^2\cdot\cos v\cdot 4=12\sin^2 4x\cos 4x.$$

Find the derivatives of the following functions (the rule for differentiating a composite function is not used in problems 368-408).

A. Algebraic Functions

368. $y=x^5-4x^3+2x-3.$

369. $y=\frac{1}{4}-\frac{1}{3}x+x^2-0.5x^4.$

370. $y=ax^2+bx+c.$

371. $y=\frac{-5x^3}{a}.$

372. $y=at^m+bt^{m+n}.$

373. $y=\frac{ax^6+b}{\sqrt{a^2+b^2}}.$

374. $y=\frac{\pi}{x}+\ln 2.$

375. $y=3x^{\frac{2}{3}}-2x^{\frac{5}{2}}+x^{-3}.$

376*. $y=x^2\sqrt[3]{x^2}.$

377. $y=\frac{a}{\sqrt[3]{x^2}}-\frac{b}{x\sqrt[3]{x}}.$

378. $y=\frac{a+bx}{c+dx}.$

379. $y=\frac{2x+3}{x^2-5x+5}.$

380. $y=\frac{2}{2x-1}-\frac{1}{x}.$

381. $y=\frac{1+\sqrt{z}}{1-\sqrt{z}}$

B. Inverse Circular and Trigonometric Functions

382. $y=5\sin x+3\cos x.$

383. $y=\tan x-\cot x.$

384. $y=\frac{\sin x+\cos x}{\sin x-\cos x}.$

385. $y=2t\sin t-(t^2-2)\cos t.$

386. $y=\operatorname{arc\,tan} x+\operatorname{arc\,cot} x.$

387. $y=x\cot x.$

388. $y=x\operatorname{arc\,sin} x.$

389. $y=\frac{(1+x^2)\operatorname{arc\,tan} x-x}{2}.$

C. Exponential and Logarithmic Functions

390. $y = x^7 \cdot e^x$.

391. $y = (x-1)e^x$.

392. $y = \frac{e^x}{x^2}$.

393. $y = \frac{x^5}{e^x}$.

394. $f(x) = e^x \cos x$.

395. $y = (x^2 - 2x + 2)e^x$.

396. $y = e^x \arcsin x$.

397. $y = \frac{x^2}{\ln x}$.

398. $y = x^3 \ln x - \frac{x^3}{3}$.

399. $y = \frac{1}{x} + 2\ln x - \frac{\ln x}{x}$.

400. $y = \ln x \log x - \ln a \log_a x$.

D. Hyperbolic and Inverse Hyperbolic Functions

401. $y = x \sinh x$.

402. $y = \frac{x^2}{\cosh x}$.

403. $y = \tanh x - x$.

404 $y = \frac{3\coth x}{\ln x}$

405. $y = \arctan x - \operatorname{arc\,tanh} x$.

406. $y = \arcsin x \operatorname{arc\,sinh} x$.

407. $y = \frac{\operatorname{arc\,cosh} x}{x}$.

408. $y = \frac{\operatorname{arc\,coth} x}{1-x^2}$.

E. Composite Functions

In problems 409 to 466, use the rule for differentiating a composite function with one intermediate argument.

Find the derivatives of the following functions:

409**. $y = (1 + 3x - 5x^2)^{30}$.

Solution. Denote $1 + 3x - 5x^2 = u$; then $y = u^{30}$. We have:

$$y'_u = 30u^{29}; \quad u'_x = 3 - 10x;$$

$$y'_x = 30u^{29} \cdot (3 - 10x) = 30(1 + 3x - 5x^2)^{29} \cdot (3 - 10x).$$

410. $y = \left(\frac{ax+b}{c}\right)^3$.

411. $f(y) = (2a + 3by)^2$.

412. $y = (3 + 2x^2)^4$.

413. $y = \frac{3}{56(2x-1)^7} - \frac{1}{24(2x-1)^6} - \frac{1}{40(2x-1)^5}$.

414. $y = \sqrt{1-x^2}$.

415. $y = \sqrt[3]{a + bx^3}$.

416. $y = (a^{2/3} - x^{2/3})^{3/2}$.

417. $y=(3-2\sin x)^5$.

Solution. $y'=5(3-2\sin x)^4\cdot(3-2\sin x)'=5(3-2\sin x)^4(-2\cos x)=-10\cos x(3-2\sin x)^4$.

418. $y=\tan x-\frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x$.

419. $y=\sqrt{\cot x}-\sqrt{\cot \alpha}$.

420. $y=2x+5\cos^3 x$.

421*. $x=\operatorname{cosec}^2 t+\sec^2 t$.

422. $f(x)=-\frac{1}{6(1-3\cos x)^2}$.

423. $y=\frac{1}{3\cos^3 x}-\frac{1}{\cos x}$.

424. $y=\sqrt{\frac{3\sin x-2\cos x}{5}}$.

425. $y=\sqrt[3]{\sin^2 x}+\frac{1}{\cos^3 x}$.

426. $y=\sqrt{1+\arcsin x}$.

427. $y=\sqrt{\arctan x}-(\arcsin x)^3$.

428. $y=\frac{1}{\arctan x}$.

429. $y=\sqrt{xe^x+x}$.

430. $y=\sqrt[3]{2e^x-2^x+1}+\ln^5 x$.

431. $y=\sin 3x+\cos\frac{x}{5}+\tan\sqrt{x}$.

Solution. $y'=\cos 3x\cdot(3x)'-\sin\frac{x}{5}\left(\frac{x}{5}\right)'+\frac{1}{\cos^2\sqrt{x}}(\sqrt{x})'=3\cos 3x-\frac{1}{5}\sin\frac{x}{5}+\frac{1}{2\sqrt{x}\cos^2\sqrt{x}}$.

432. $y=\sin(x^2-5x+1)+\tan\frac{a}{x}$.

433. $f(x)=\cos(\alpha x+\beta)$.

434. $f(t)=\sin t\sin(t+\varphi)$.

435. $y=\frac{1+\cos 2x}{1-\cos 2x}$.

436. $f(x)=a\cot\frac{x}{a}$.

437. $y=-\frac{1}{20}\cos(5x^2)-\frac{1}{4}\cos x^2$.

438. $y=\arcsin 2x$.

Solution. $y'=\frac{1}{\sqrt{1-(2x)^2}}\cdot(2x)'=\frac{2}{\sqrt{1-4x^2}}$.

439. $y=\arcsin\frac{1}{x^2}$.

440. $f(x)=\arccos\sqrt{x}$.

441. $y=\arctan\frac{1}{x}$.

442. $y=\operatorname{arccot}\frac{1+x}{1-x}$.

443. $y = 5e^{-x^2}$.

444. $y = \frac{1}{5^{x^2}}$.

445. $y = x^2\, 10^{2x}$.

446. $f(t) = t \sin 2^t$.

447. $y = \arccos e^x$.

448. $y = \ln(2x+7)$.

449. $y = \log \sin x$.

450. $y = \ln(1 - x^2)$.

451. $y = \ln^2 x - \ln(\ln x)$.

452. $y = \ln(e^x + 5 \sin x - 4 \arcsin x)$.

453. $y = \arctan(\ln x) + \ln(\arctan x)$.

454. $y = \sqrt{\ln x + 1} + \ln(\sqrt{x} + 1)$.

F. Miscellaneous Functions

455**. $y = \sin^3 5x \cos^2 \frac{x}{3}$.

456. $y = -\frac{11}{2(x-2)^2} - \frac{4}{x-2}$.

457. $y = -\frac{15}{4(x-3)^4} - \frac{10}{3(x-3)^3} - \frac{1}{2(x-3)^2}$.

458. $y = \frac{x^8}{8(1-x^2)^4}$.

459. $y = \frac{\sqrt{2x^2 - 2x + 1}}{x}$.

460. $y = \frac{x}{a^2 \sqrt{a^2 + x^2}}$.

461. $y = \frac{x^3}{3\sqrt{(1+x^2)^3}}$.

462. $y = \frac{3}{2} \sqrt[3]{x^2} + \frac{18}{7} x \sqrt[6]{x} + \frac{9}{5} x \sqrt[3]{x^2} + \frac{6}{13} x^2 \sqrt[6]{x}$.

463. $y = \frac{1}{8} \sqrt[3]{(1+x^3)^8} - \frac{1}{5} \sqrt[3]{(1+x^3)^5}$.

464. $y = \frac{4}{3} \sqrt[4]{\frac{x-1}{x+2}}$.

465. $y = x^4 (a - 2x^3)^2$.

466. $y = \left(\frac{a + bx^n}{a - bx^n}\right)^m$.

467. $y = \frac{9}{5(x+2)^5} - \frac{3}{(x+2)^4} + \frac{2}{(x+2)^3} - \frac{1}{2(x+2)^2}$.

468. $y = (a+x)\sqrt{a-x}$.

469. $y = \sqrt{(x+a)(x+b)(x+c)}$.

470. $z = \sqrt[3]{y + \sqrt{y}}$.

471. $f(t) = (2t+1)(3t+2)\sqrt[3]{3t+2}$.

472. $x=\frac{1}{\sqrt{2ay-y^2}}$.

473. $y=\ln(\sqrt{1+e^x}-1)-\ln(\sqrt{1+e^x}+1)$.

474. $y=\frac{1}{15}\cos^3 x(3\cos^2 x-5)$.

475. $y=\frac{(\tan^2 x-1)(\tan^4 x+10\tan^2 x+1)}{3\tan^3 x}$.

476. $y=\tan^5 5x$.

477. $y=\frac{1}{2}\sin(x^2)$.

478. $y=\sin^2(t^3)$.

479. $y=3\sin x\cos^2 x+\sin^3 x$.

480. $y=\frac{1}{3}\tan^3 x-\tan x+x$.

481. $y=-\frac{\cos x}{3\sin^3 x}+\frac{4}{3}\cot x$.

482. $y=\sqrt{\alpha\sin^2 x+\beta\cos^2 x}$.

483. $y=\arcsin x^2+\arccos x^2$.

484. $y=\frac{1}{2}(\arcsin x)^2\arccos x$.

485. $y=\arcsin\frac{x^2-1}{x^2}$.

486. $y=\arcsin\frac{x}{\sqrt{1+x^2}}$.

487. $y=\frac{\arccos x}{\sqrt{1-x^2}}$.

488. $y=\frac{1}{\sqrt{b}}\arcsin\left(x\sqrt{\frac{b}{a}}\right)$.

489. $y=\sqrt{a^2-x^2}+a\arcsin\frac{x}{a}$.

490. $y=x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}$.

491. $y=\arcsin(1-x)+\sqrt{2x-x^2}$.

492. $y=\left(x-\frac{1}{2}\right)\arcsin\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}$.

493. $y=\ln(\arcsin 5x)$.

494. $y=\arcsin(\ln x)$.

495. $y=\arctan\frac{x\sin\alpha}{1-x\cos\alpha}$.

496. $y=\frac{2}{3}\arctan\frac{5\tan\frac{x}{2}+4}{3}$.

497. $y=3b^2\arctan\sqrt{\frac{x}{b-x}}-(3b+2x)\sqrt{bx-x^2}$.

498. $y=-\sqrt{2}\operatorname{arccot}\frac{\tan x}{\sqrt{2}}-x$.

499. $y=\sqrt{e^{ax}}$.

500. $y=e^{\sin^2 x}$.

501. $F(x)=(2ma^{mx}+b)^p$.

502. $F(t)=e^{\alpha t}\cos\beta t$.

503. $y=\frac{(\alpha\sin\beta x-\beta\cos\beta x)e^{\alpha x}}{\alpha^2+\beta^2}$.

504. $y=\frac{1}{10}e^{-x}(3\sin 3x-\cos 3x)$.

505. $y=x^n a^{-x^2}$.

506. $y=\sqrt{\cos x}\,a^{\sqrt{\cos x}}$.

507. $y=3^{\cot\frac{1}{x}}$.

508. $y=\ln(ax^2+bx+c)$.

509. $y=\ln(x+\sqrt{a^2+x^2})$.

510. $y=x-2\sqrt{x}+2\ln(1+\sqrt{x})$.

511. $y=\ln(a+x+\sqrt{2ax+x^2})$.

512. $y=\frac{1}{\ln^2 x}$.

513. $y=\ln\cos\frac{x-1}{x}$.

514*. $y=\ln\frac{(x-2)^5}{(x+1)^3}$.

515. $y=\ln\frac{(x-1)^3(x-2)}{x-3}$.

516. $y=-\frac{1}{2\sin^2 x}+\ln\tan x$.

517. $y=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})$.

518. $y=\ln\ln(3-2x^3)$.

519. $y=5\ln^3(ax+b)$.

520. $y=\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}$.

521. $y=\frac{m}{2}\ln(x^2-a^2)+\frac{n}{2a}\ln\frac{x-a}{x+a}$.

522. $y=x\cdot\sin\left(\ln x-\frac{\pi}{4}\right)$.

523. $y=\frac{1}{2}\ln\tan\frac{x}{2}-\frac{1}{2}\frac{\cos x}{\sin^2 x}$.

524. $f(x)=\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}$.

525. $y=\frac{1}{3}\ln\frac{x^2-2x+1}{x^2+x+1}$.

526. $y=2^{\arcsin 3x}+(1-\arccos 3x)^2$.

527. $y=3^{\frac{\sin ax}{\cos bx}}+\frac{1}{3}\frac{\sin^3 ax}{\cos^3 bx}$.

528. $y=\frac{1}{\sqrt{3}}\ln\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}$.

529. $y=\arctan\ln x$.

530. $y=\ln\arcsin x+\frac{1}{2}\ln^2 x+\arcsin\ln x$.

531. $y=\arctan\ln\frac{1}{x}$.

532. $y=\frac{\sqrt{2}}{3}\arctan\frac{x}{\sqrt{2}}+\frac{1}{6}\ln\frac{x-1}{x+1}$.

533. $y=\ln\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}+2\arctan\sqrt{\sin x}$.

534. $y=\frac{3}{4}\ln\frac{x^2+1}{x^2-1}+\frac{1}{4}\ln\frac{x-1}{x+1}+\frac{1}{2}\arctan x$.

535. $f(x)=\frac{1}{2}\ln(1+x)-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$.

536. $f(x)=\frac{x\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{1-x^2}$.

537. $y=\sinh^2 2x$.

538. $y=e^{\alpha x}\cosh\beta x$.

539. $y=\tanh^2 2x$.

540. $y=\ln\sinh 2x$.

541. $y=\operatorname{arc\,sinh}\frac{x^2}{a^2}$.

542. $y=\operatorname{arc\,cosh}\ln x$.

543. $y=\operatorname{arc\,tanh}(\tan x)$.

544. $y=\operatorname{arc\,coth}(\sec x)$.

545. $y=\operatorname{arc\,tanh}\frac{2x}{1+x^2}$.

546. $y=\frac{1}{2}(x^2-1)\operatorname{arc\,tanh} x+\frac{1}{2}x$.

547. $y=\left(\frac{1}{2}x^2+\frac{1}{4}\right)\operatorname{arc\,sinh} x-\frac{1}{4}x\sqrt{1+x^2}$.

548. Find y', if:

a) $y=|x|$;

b) $y=x|x|$.

Construct the graphs of the functions y and y'.

549. Find y' if

$$y=\ln|x| \quad (x\neq 0).$$

550. Find $f'(x)$ if

$$f(x)=\begin{cases} 1-x & \text{for } x\leqslant 0, \\ e^{-x} & \text{for } x>0. \end{cases}$$

551. Calculate $f'(0)$ if

$$f(x)=e^{-x}\cos 3x.$$

Solution. $f'(x)=e^{-x}(-3\sin 3x)-e^{-x}\cos 3x$;

$f'(0)=e^0(-3\sin 0)-e^0\cos 0=-1$.

552. $f(x)=\ln(1+x)+\arcsin\frac{x}{2}$. Find $f'(1)$.

553. $y=\tan^3\frac{\pi x}{6}$. Find $\left(\frac{dy}{dx}\right)_{x=2}$.

554. Find $f'_+(0)$ and $f'_-(0)$ of the functions:

a) $f(x)=\sqrt{\sin(x^2)}$;

b) $f(x)=\arcsin\frac{a^2-x^2}{a^2+x^2}$;

c) $f(x)=\frac{x}{1+e^{\frac{1}{x}}}$, $x\neq 0$; $f(0)=0$;

d) $f(x)=x^2\sin\frac{1}{x}$, $x\neq 0$; $f(0)=0$;

e) $f(x)=x\sin\frac{1}{x}$ $x\neq 0$; $f(0)=0$

555. Find $f(0)+xf'(0)$ of the function $f(x)=e^{-x}$.

556. Find $f(3)+(x-3)f'(3)$ of the function $f(x)=\sqrt{1+x}$.

557. Given the functions $f(x)=\tan x$ and $\varphi(x)=\ln(1-x)$, find $\frac{f'(0)}{\varphi'(0)}$.

558. Given the functions $f(x)=1-x$ and $\varphi(x)=1-\sin\frac{\pi x}{2}$, find $\frac{\varphi'(1)}{f'(1)}$.

559. Prove that the derivative of an even function is an odd function, and the derivative of an odd function is an even function.

560. Prove that the derivative of a periodic function is also a periodic function.

561. Show that the function $y=xe^{-x}$ satisfies the equation $xy'=(1-x)y$.

562. Show that the function $y=xe^{-\frac{x^2}{2}}$ satisfies the equation $xy'=(1-x^2)y$.

563. Show that the function $y=\frac{1}{1+x+\ln x}$ satisfies the equation $xy'=y(y\ln x-1)$.

G. Logarithmic Derivative

A *logarithmic derivative* of a function $y=f(x)$ is the derivative of the logarithm of this function; that is,

$$(\ln y)'=\frac{y'}{y}=\frac{f'(x)}{f(x)}$$

Finding the derivative is sometimes simplified by first taking logs of the function.

Example. Find the derivative of the exponential function

$$y=u^v,$$

where $u=\varphi(x)$ and $v=\psi(x)$.

Solution. Taking logarithms we get

$$\ln y=v\ln u.$$

Differentiate both sides of this equation with respect *to* x:

$$(\ln y)'=v'\ln u+v(\ln u)',$$

or

$$\frac{1}{y}y'=v'\ln u+v\frac{1}{u}u',$$

whence

$$y'=y\left(v'\ln u+\frac{v}{u}u'\right),$$

or

$$y'=u^v\left(v'\ln u+\frac{v}{u}u'\right)$$

564. Find y', if

$$y=\sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3 x\cos^2 x.$$

Solution. $\ln y=\frac{2}{3}\ln x+\ln(1-x)-\ln(1+x^2)+3\ln\sin x+2\ln\cos x;$

$$\frac{1}{y}y'=\frac{2}{3}\,\frac{1}{x}+\frac{(-1)}{1-x}-\frac{2x}{1+x^2}+3\frac{1}{\sin x}\cos x-\frac{2\sin x}{\cos x},$$

whence $y'=y\left(\frac{2}{3x}-\frac{1}{1-x}-\frac{2x}{1+x^2}+3\cot x-2\tan x\right).$

565. Find y', if $y=(\sin x)^x$.

Solution. $\ln y=x\ln\sin x;\quad \frac{1}{y}y'=\ln\sin x+x\cot x;$

$y'=(\sin x)^x(\ln\sin x+x\cot x).$

In the following problems find y' after first taking logs of the function $y=f(x)$:

566. $y=(x+1)(2x+1)(3x+1)$.

567. $y=\frac{(x+2)^2}{(x+1)^3(x+3)^4}$.

568. $y=\sqrt{\frac{x(x-1)}{x-2}}$.

569. $y=x\sqrt[3]{\frac{x^2}{x^2+1}}$.

570. $y=\frac{(x-2)^9}{\sqrt{(x-1)^5(x-3)^{11}}}$.

571. $y=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\sqrt{(x+3)^3}}$.

572. $y=x^x$.

573. $y=x^{x^2}$.

574. $y=\sqrt[x]{x}$.

575. $y=x^{\sqrt{x}}$.

576. $y=x^{x^x}$.

577. $y=x^{\sin x}$.

578. $y=(\cos x)^{\sin x}$.

579. $y=\left(1+\frac{1}{x}\right)^x$.

580. $y=(\arctan x)^x$.

Sec. 3. The Derivatives of Functions Not Represented Explicitly

1°. The derivative of an inverse function. If a function $y=f(x)$ has a derivative $y'_x\neq 0$, then the derivative of the inverse function $x=f^{-1}(y)$ is

$$x'_y=\frac{1}{y'_x}$$

or

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

Example 1. Find the derivative x'_y, if

$$y=x+\ln x.$$

Solution. We have $y'_x=1+\frac{1}{x}=\frac{x+1}{x}$; hence, $x'_y=\frac{x}{x+1}$.

2°. **The derivatives of functions represented parametrically.** If a function y is related to an argument x by means of a parameter t,

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$$

then

$$y'_x=\frac{y'_t}{x'_t},$$

or, in other notation,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Example 2. Find $\frac{dy}{dx}$, if

$$\left.\begin{array}{l} x=a\cos t, \\ y=a\sin t \end{array}\right\}$$

Solution. We find $\frac{dx}{dt}=-a\sin t$ and $\frac{dy}{dx}=a\cos t$. Whence

$$\frac{dy}{dx}=-\frac{a\cos t}{a\sin t}=-\cot t.$$

3°. **The derivative of an implicit function.** If the relationship between x and y is given in implicit form,

$$F(x, y)=0, \tag{1}$$

then to find the derivative $y'_x=y'$ in the simplest cases it is sufficient: 1) to calculate the derivative, with respect to x, of the left side of equation (1), taking y as a function of x; 2) to equate this derivative to zero, that is, to put

$$\frac{d}{dx}F(x, y)=0, \tag{2}$$

and 3) to solve the resulting equation for y'.

Example 3. Find the derivative y'_x if

$$x^3+y^3-3axy=0. \tag{3}$$

Solution. Forming the derivative of the left side of (3) and equating it to zero, we get

$$3x^2+3y^2y'-3a(y+xy')=0,$$

whence

$$y'=\frac{x^2-ay}{ax-y^2}.$$

581. Find the derivative x'_y if

a) $y=3x+x^3$;

b) $y=x-\frac{1}{2}\sin x$;

c) $y=0.1x+e^{\frac{x}{2}}$.

In the following problems, find the derivative $y'=\frac{dy}{dx}$ of the functions y represented parametrically:

582. $\begin{cases} x=2t-1, \\ y=t^3. \end{cases}$

583. $\begin{cases} x=\dfrac{1}{t+1}, \\ y=\left(\dfrac{t}{t+1}\right)^2. \end{cases}$

584. $\begin{cases} x=\dfrac{2at}{1+t^2}, \\ y=\dfrac{a(1-t^2)}{1+t^2}. \end{cases}$

585. $\begin{cases} x=\dfrac{3at}{1+t^3}, \\ y=\dfrac{3at^2}{1+t^3}. \end{cases}$

586. $\begin{cases} x=\sqrt{t}, \\ y=\sqrt[3]{t}. \end{cases}$

587. $\begin{cases} x=\sqrt{t^2+1}, \\ y=\dfrac{t-1}{\sqrt{t^2+1}}. \end{cases}$

588. $\begin{cases} x=a(\cos t+t\sin t), \\ y=a(\sin t-t\cos t). \end{cases}$

589. $\begin{cases} x=a\cos^2 t, \\ y=b\sin^2 t. \end{cases}$

590. $\begin{cases} x=a\cos^3 t, \\ y=b\sin^3 t. \end{cases}$

591. $\begin{cases} x=\dfrac{\cos^3 t}{\sqrt{\cos 2t}}, \\ y=\dfrac{\sin^3 t}{\sqrt{\cos 2t}}. \end{cases}$

592. $\begin{cases} x=\operatorname{arc\,cos}\dfrac{1}{\sqrt{1+t^2}}, \\ y=\operatorname{arc\,sin}\dfrac{t}{\sqrt{1+t^2}}. \end{cases}$

593. $\begin{cases} x=e^{-t}, \\ y=e^{2t}. \end{cases}$

594. $\begin{cases} x=a\left(\ln\tan\dfrac{t}{2}+\cos t-\sin t\right), \\ y=a(\sin t+\cos t). \end{cases}$

595. Calculate $\frac{dy}{dx}$ when $t=\frac{\pi}{2}$ if

$$\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t). \end{cases}$$

Solution. $\frac{dy}{dx}=\frac{a\sin t}{a(1-\cos t)}=\frac{\sin t}{1-\cos t}$

and

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}}=\frac{\sin\frac{\pi}{2}}{1-\cos\frac{\pi}{2}}=1.$$

596. Find $\frac{dy}{dx}$ when $t=1$ if $\begin{cases} x=t\ln t, \\ y=\frac{\ln t}{t}. \end{cases}$

597. Find $\frac{dy}{dx}$ when $t=\frac{\pi}{4}$ if $\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$

598. Prove that a function y represented parametrically by the equations

$$\begin{cases} x=2t+3t^2, \\ y=t^2+2t^3, \end{cases}$$

satisfies the equation

$$y=\left(\frac{dy}{dx}\right)^2+2\left(\frac{dy}{dx}\right)^3.$$

599. When $x=2$ the following equation is true:

$$x^2=2x.$$

Does it follow from this that

$$(x^2)'=(2x)'$$

when $x=2$?

600. Let $y=\sqrt{a^2-x^2}$. Is it possible to perform term-by-term differentiation of

$$x^2+y^2=a^2?$$

In the examples that follow it is required to find the derivative $y'=\frac{dy}{dx}$ of the implicit functions y.

601. $2x-5y+10=0.$

602. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

603. $x^3+y^3=a^3.$

604. $x^3+x^2y+y^2=0.$

605. $\sqrt{x}+\sqrt{y}=\sqrt{a}.$

606. $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}.$

607. $y^3=\frac{x-y}{x+y}.$

608. $y-0.3\sin y=x.$

609. $a\cos^2(x+y)=b.$

610. $\tan y=xy.$

611. $xy=\arctan\frac{x}{y}.$

612. $\arctan(x+y)=x.$

613. $e^y=x+y.$

614. $\ln x+e^{-\frac{y}{x}}=c.$

615. $\ln y+\frac{x}{y}=c.$

616. $\arctan\frac{y}{x}=\frac{1}{2}\ln(x^2+y^2).$

617. $\sqrt{x^2+y^2}=c\arctan\frac{y}{x}$. **618.** $x^y=y^x$.

619. Find y' at the point $M(1,1)$, if

$$2y=1+xy^3.$$

Solution. Differentiating, we get $2y'=y^3+3xy^2y'$. Putting $x=1$ and $y=1$, we obtain $2y'=1+3y'$, whence $y'=-1$.

620. Find the derivatives y' of specified functions y at the indicated points:

a) $(x+y)^3=27(x-y)$ for $x=2$ and $y=1$;
b) $ye^y=e^{x+1}$ for $x=0$ and $y=1$;
c) $y^2=x+\ln\frac{y}{x}$ for $x=1$ and $y=1$.

Sec. 4. Geometrical and Mechanical Applications of the Derivative

1°. Equations of the tangent and the normal. From the geometric significance of a derivative it follows that the *equation of the tangent* to a curve $y=f(x)$ or $F(x,y)=0$ at a point $M(x_0, y_0)$ will be

$$y-y_0=y_0'(x-x_0),$$

where y_0' is the value of the derivative y' at the point $M(x_0, y_0)$. The straight line passing through the point of tangency perpendicularly to the tangent is called the *normal to the curve*. For the normal we have the equation

$$x-x_0+y_0'(y-y_0)=0.$$

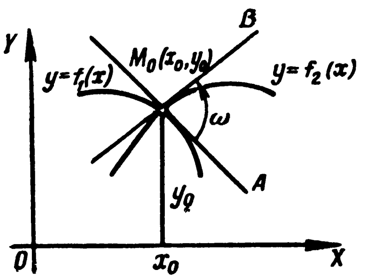

Fig. 12

2°. The angle between curves. The angle between the curves

$$y=f_1(x)$$

and

$$y=f_2(x)$$

at their common point $M_0(x_0, y_0)$ (Fig. 12) is the angle ω between the tangents M_0A and M_0B to these curves at the point M_0.

Using a familiar formula of analytic geometry, we get

$$\tan\omega=\frac{f_2'(x_0)-f_1'(x_0)}{1+f_1'(x_0)\cdot f_2'(x_0)}.$$

3°. Segments associated with the tangent and the normal in a rectangular coordinate system. The tangent and the normal determine the following four segments (Fig. 13):

$t=TM$ is the so-called *segment of the tangent*,
$S_t=TK$ is the *subtangent*,
$n=NM$ is the *segment of the normal*,
$S_n=KN$ is the *subnormal*.

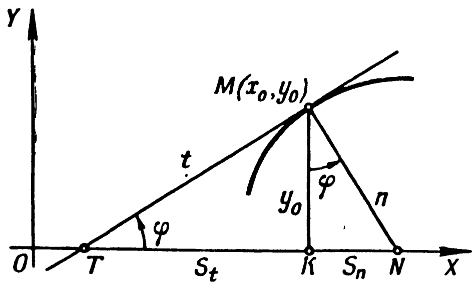

Fig. 13

Since $KM=|y_0|$ and $\tan\varphi=y_0'$, it follows that

$$t=TM=\left|\frac{y_0}{y_0'}\sqrt{1+(y_0')^2}\right|;\quad n=NM=\left|y_0\sqrt{1+(y_0')^2}\right|;$$

$$S_t=TK=\left|\frac{y_0}{y_0'}\right|;\quad S_n=\left|y_0y_0'\right|.$$

4°. Segments associated with the tangent and the normal in a polar system of coordinates. If a curve is given in polar coordinates by the equation $r=f(\varphi)$, then the angle μ formed by the tangent MT and the radius vector $r=OM$ (Fig. 14), is defined by the following formula:

$$\tan\mu=r\frac{d\varphi}{dr}=\frac{r}{r'}.$$

The tangent MT and the normal MN at the point M together with the radius vector of the point of tangency and with the perpendicular to the radius vector drawn through the pole O determine the following four segments (see Fig. 14):

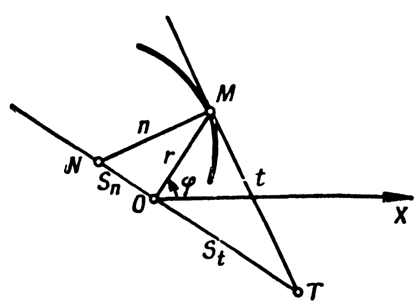

Fig. 14

$t=MT$ is the *segment of the polar tangent*,
$n=MN$ is the *segment of the polar normal*,
$S_t=OT$ is the *polar subtangent*,
$S_n=ON$ is the *polar subnormal*.

These segments are expressed by the following formulas:

$$t = MT = \frac{r}{|r'|}\sqrt{r^2+(r')^2}; \qquad S_t = OT = \frac{r^2}{|r'|};$$

$$n = MN = \sqrt{r^2+(r')^2}; \qquad S_n = ON = |r'|.$$

621. What angles φ are formed with the x-axis by the tangents to the curve $y = x - x^2$ at points with abscissas:

a) $x=0$; b) $x=1/2$; c) $x=1$?

Solution. We have $y' = 1-2x$. Whence a) $\tan\varphi = 1$, $\varphi = 45°$; b) $\tan\varphi = 0$, $\varphi = 0°$; c) $\tan\varphi = -1$, $\varphi = 135°$ (Fig. 15).

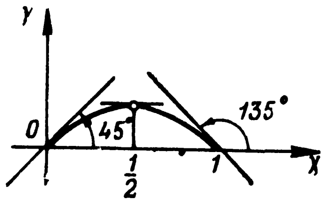

Fig. 15

622. At what angles do the sine curves $y = \sin x$ and $y = \sin 2x$ intersect the axis of abscissas at the origin?

623. At what angle does the tangent curve $y = \tan x$ intersect the axis of abscissas at the origin?

624. At what angle does the curve $y = e^{0.5x}$ intersect the straight line $x = 2$?

625. Find the points at which the tangents to the curve $y = 3x^4 + 4x^3 - 12x^2 + 20$ are parallel to the x-axis.

626. At what point is the tangent to the parabola

$$y = x^2 - 7x + 3$$

parallel to the straight line $5x + y - 3 = 0$?

627. Find the equation of the parabola $y = x^2 + bx + c$ that is tangent to the straight line $x = y$ at the point (1,1).

628. Determine the slope of the tangent to the curve $x^3 + y^3 - xy - 7 = 0$ at the point (1,2).

629. At what point of the curve $y^2 = 2x^3$ is the tangent perpendicular to the straight line $4x - 3y + 2 = 0$?

630. Write the equation of the tangent and the normal to the parabola

$$y = \sqrt{x}$$

at the point with abscissa $x = 4$.

Solution. We have $y' = \frac{1}{2\sqrt{x}}$; whence the slope of the tangent is $k = [y']_{x=4} = \frac{1}{4}$. Since the point of tangency has coordinates $x = 4$, $y = 2$, it follows that the equation of the tangent is $y - 2 = 1/4\,(x-4)$ or $x - 4y + 4 = 0$.

Since the slope of the normal must be perpendicular,

$$k_1 = -4;$$

whence the equation of the normal: $y - 2 = -4(x-4)$ or $4x + y - 18 = 0$.

631. Write the equations of the tangent and the normal to the curve $y=x^3+2x^2-4x-3$ at the point $(-2,5)$.

632. Find the equations of the tangent and the normal to the curve

$$y=\sqrt[3]{x-1}$$

at the point (1,0).

633. Form the equations of the tangent and the normal to the curves at the indicated points:

a) $y=\tan 2x$ at the origin;

b) $y=\arcsin\frac{x-1}{2}$ at the point of intersection with the x-axis;

c) $y=\arccos 3x$ at the point of intersection with the y-axis;

d) $y=\ln x$ at the point of intersection with the x-axis;

e) $y=e^{1-x^2}$ at the points of intersection with the straight line $y=1$.

634. Write the equations of the tangent and the normal at the point (2,2) to the curve

$$x=\frac{1+t}{t^3},$$

$$y=\frac{3}{2t^2}+\frac{1}{2t}.$$

635. Write the equations of the tangent to the curve

$$x=t\cos t,\quad y=t\sin t$$

at the origin and at the point $t=\frac{\pi}{4}$.

636. Write the equations of the tangent and the normal to the curve $x^3+y^2+2x-6=0$ at the point with ordinate $y=3$.

637. Write the equation of the tangent to the curve $x^5+y^5-2xy=0$ at the point (1,1).

638. Write the equations of the tangents and the normals to the curve $y=(x-1)(x-2)(x-3)$ at the points of its intersection with the x-axis.

639. Write the equations of the tangent and the normal to the curve $y^4=4x^4+6xy$ at the point (1,2).

640*. Show that the segment of the tangent to the hyperbola $xy=a^2$ (the segment lies between the coordinate axes) is divided in two at the point of tangency.

641. Show that in the case of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ the segment of the tangent between the coordinate axes has a constant value equal to a.

642. Show that the normals to the involute of the circle

$$x=a(\cos t+t\sin t),\quad y=a(\sin t-t\cos t)$$

are tangents to the circle $x^2+y^2=a^2$.

643. Find the angle at which the parabolas $y=(x-2)^2$ and $y=-4+6x-x^2$ intersect.

644. At what angle do the parabolas $y=x^2$ and $y=x^3$ intersect?

645. Show that the curves $y=4x^2+2x-8$ and $y=x^3-x+10$ are tangent to each other at the point (3,34). Will we have the same thing at (—2,4)?

646. Show that the hyperbolas

$$xy=a^2;\quad x^2-y=b^2$$

intersect at a right angle.

647. Given a parabola $y^2=4x$. At the point (1,2) evaluate the lengths of the segments of the subtangent, subnormal, tangent, and normal.

648. Find the length of the segment of the subtangent of the curve $y=2^x$ at any point of it.

649. Show that in the equilateral hyperbola $x^2-y^2=a^2$ the length of the normal at any point is equal to the radius vector of this point.

650. Show that the length of the segment of the subnormal in the hyperbola $x^2-y^2=a^2$ at any point is equal to the abscissa of this point.

651. Show that the segments of the subtangents of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=a^2$ at points with the same abscissas are equal. What procedure of construction of the tangent to the ellipse follows from this?

652. Find the length of the segment of the tangent, the normal, the subtangent, and the subnormal of the cycloid

$$\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t) \end{cases}$$

at an arbitrary point $t=t_0$.

653. Find the angle between the tangent and the radius vector of the point of tangency in the case of the logarithmic spiral

$$r=ae^{k\varphi}.$$

654. Find the angle between the tangent and the radius vector of the point of tangency in the case of the lemniscate $r^2=a^2\cos 2\varphi$.

655. Find the lengths of the segments of the polar subtangent, subnormal, tangent and normal, and also the angle between the tangent and the radius vector of the point of tangency in the case of the spiral of Archimedes

$$r = a\varphi$$

at a point with polar angle $\varphi = 2\pi$.

656. Find the lengths of the segments of the polar subtangent, subnormal, tangent, and normal, and also the angle between the tangent and the radius vector in the hyperbolic spiral $r = \frac{a}{\varphi}$ at an arbitrary point $\varphi = \varphi_0$; $r = r_0$.

657. The law of motion of a point on the x-axis is

$$x = 3t - t^3.$$

Find the velocity of the point at $t_0 = 0$, $t_1 = 1$, and $t_2 = 2$ (x is in centimetres and t is in seconds).

658. Moving along the x-axis are two points that have the following laws of motion: $x = 100 + 5t$ and $x = 1/2t^2$, where $t \geqslant 0$. With what speed are these points receding from each other at the time of encounter (x is in centimetres and t is in seconds)?

659. The end-points of a segment $AB = 5$ m are sliding along the coordinate axes OX and OY (Fig. 16). A is moving at 2 m/sec.

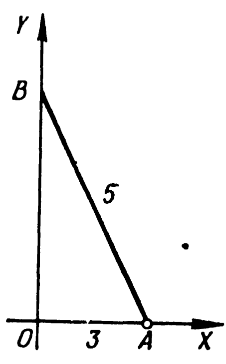

Fig. 16

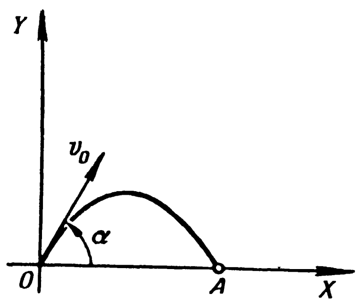

Fig. 17

What is the rate of motion of B when A is at a distance $OA = 3$ m from the origin?

660*. The law of motion of a material point thrown up at an angle α to the horizon with initial velocity v_0 (in the vertical plane OXY in Fig. 17) is given by the formulas (air resistance is disregarded):

$$x = v_0 t \cos \alpha, \qquad y = v_0 t \sin \alpha - \frac{gt^2}{2},$$

where t is the time and g is the acceleration of gravity. Find the trajectory of motion and the distance covered. Also determine the speed of motion and its direction.

661. A point is in motion along a hyperbola $y = \frac{10}{x}$ so that its abscissa x increases uniformly at a rate of 1 unit per second. What is the rate of change of its ordinate when the point passes through (5,2)?

662. At what point of the parabola $y^2 = 18x$ does the ordinate increase at twice the rate of the abscissa?

663. One side of a rectangle, $a = 10$ cm, is of constant length, while the other side, b, increases at a constant rate of 4 cm/sec. At what rate are the diagonal of the rectangle and its area increasing when $b = 30$ cm?

664. The radius of a sphere is increasing at a uniform rate of 5 cm/sec. At what rate are the area of the surface of the sphere and the volume of the sphere increasing when the radius becomes 50 cm?

665. A point is in motion along the spiral of Archimedes

$$r = a\varphi$$

($a = 10$ cm) so that the angular velocity of rotation of its radius vector is constant and equal to 6° per second. Determine the rate of elongation of the radius vector r when $r = 25$ cm.

666. A nonhomogeneous rod AB is 12 cm long. The mass of a part of it, AM, increases with the square of the distance of the moving point M from the end A and is 10 gm when $AM = 2$ cm. Find the mass of the entire rod AB and the linear density at any point M. What is the linear density of the rod at A and B?

Sec. 5. Derivatives of Higher Orders

1°. **Definition of higher derivatives.** A *derivative of the second order*, or the *second derivative*, of the function $y = f(x)$ is the derivative of its derivative; that is,

$$y'' = (y')'.$$

The second derivative may be denoted as

$$y'', \quad \text{or} \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad f''(x).$$

If $x = f(t)$ is the law of rectilinear motion of a point, then $\frac{d^2x}{dt^2}$ is the acceleration of this motion.

Generally, the nth *derivative* of a function $y=f(x)$ is the derivative of a derivative of order $(n-1)$. For the nth derivative we use the notation

$$y^{(n)}, \quad \text{or} \quad \frac{d^n y}{dx^n}, \quad \text{or} \quad f^{(n)}(x).$$

Example 1. Find the second derivative of the function

$$y=\ln(1-x).$$

Solution. $y'=\dfrac{-1}{1-x}$; $y''=\left(\dfrac{-1}{1-x}\right)'=\dfrac{1}{(1-x)^2}$.

2°. **Leibniz rule.** If the functions $u=\varphi(x)$ and $v=\psi(x)$ have derivatives up to the nth order inclusive, then to evaluate the nth derivative of a product of these functions we can use the *Leibniz rule* (or formula):

$$(uv)^{(n)}=u^{(n)}v+n\cdot u^{(n-1)}v'+\frac{n(n-1)}{1\cdot 2}u^{(-2)}v''+\ldots+uv^{(n)}.$$

3°. **Higher-order derivatives of functions represented parametrically.** If

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$$

then the derivatives $y'_x=\dfrac{dy}{dx}$, $y''_{xx}=\dfrac{d^2y}{dx^2}$, ... can successively be calculated by the formulas

$$y'_x=\frac{y'_t}{x'_t}, \quad y''_{xx}=\left(y'_x\right)'_x=\frac{\left(y'_x\right)'_t}{x'_t}, \quad y'''_{xxx}=\frac{\left(y''_{xx}\right)'_t}{x'_t} \text{ and so forth.}$$

For a second derivative we have the formula

$$y''_{xx}=\frac{x'_t y''_{tt}-x''_{tt}y'_t}{(x'_t)^3}.$$

Example 2. Find y'', if

$$\begin{cases} x=a\cos t, \\ y=b\sin t. \end{cases}$$

Solution. We have

$$y'=\frac{(b\sin t)'_t}{(a\cos t)'_t}=\frac{b\cdot\cos t}{-a\sin t}=-\frac{b}{a}\cot t.$$

and

$$y''=\frac{\left(-\frac{b}{a}\cot t\right)'_t}{(a\cos t)'_t}=\frac{-\frac{b}{a}\cdot\frac{-1}{\sin^2 t}}{-a\sin t}=-\frac{b}{a^2\sin^3 t}.$$

A. Higher-Order Derivatives of Explicit Functions

In the examples that follow, find the second derivative of th given function.

667. $y = x^8 + 7x^6 - 5x + 4$.

668. $y = e^{x^2}$.

669. $y = \sin^2 x$.

670. $y = \ln \sqrt[3]{1+x^2}$.

671. $y = \ln (x + \sqrt{a^2 + x^2})$.

672. $f(x) = (1 + x^2) \cdot \arctan x$.

673. $y = (\arcsin x)^2$.

674. $y = a \cosh \frac{x}{a}$.

675. Show that the function $y = \frac{x^2 + 2x + 2}{2}$ satisfies the differ ential equation $1 + y'^2 = 2yy''$.

676. Show that the function $y = \frac{1}{2} x^2 e^x$ satisfies the differen tial equation $y'' - 2y' + y = e^x$.

677. Show that the function $y = C_1 e^{-x} + C_2 e^{-2x}$ satisfies th equation $y'' + 3y' + 2y = 0$ for all constants C_1 and C_2.

678. Show that the function $y = e^{2x} \sin 5x$ satisfies the equa tion $y'' - 4y' + 29y = 0$.

679. Find y''', if $y = x^3 - 5x^2 + 7x - 2$.

680. Find $f'''(3)$, if $f(x) = (2x - 3)^5$.

681. Find y^{V} of the function $y = \ln (1 + x)$.

682. Find y^{VI} of the function $y = \sin 2x$.

683. Show that the function $y = e^{-x} \cos x$ satisfies the differ ential equation $y^{\mathrm{IV}} + 4y = 0$.

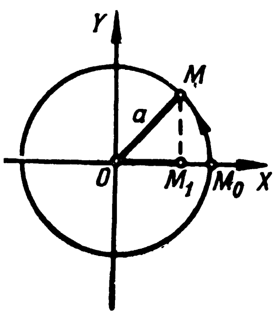

Fig. 18

684. Find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ if $f(x) = e^x \sin x$.

685. The equation of motion of a poin along the x-axis is

$$x = 100 + 5t - 0.001t^3.$$

Find the velocity and the acceleration c the point for times $t_0 = 0$, $t_1 = 1$, an $t_2 = 10$.

686. A point M is in motion around circle $x^2 + y^2 = a^2$ with constant angula velocity ω. Find the law of motion of it projection M_1 on the x-axis if at time $t =$ the point is at $M_0(a, 0)$ (Fig. 18). Find the velocity and the ac celeration of motion of M_1.

What is the velocity and the acceleration of M_1 at the in tial time and when it passes through the origin?

What are the maximum values of the absolute velocity and th absolute acceleration of M_1?

687. Find the nth derivative of the function $y=(ax+b)^n$, where n is a natural number.

688. Find the nth derivatives of the functions:

$$\text{a) } y=\frac{1}{1-x}; \quad \text{and} \quad \text{b) } y=\sqrt{x}.$$

689. Find the nth derivative of the functions:

a) $y=\sin x$; e) $y=\frac{1}{1+x}$;

b) $y=\cos 2x$; f) $y=\frac{1+x}{1-x}$;

c) $y=e^{-3x}$; g) $y=\sin^2 x$;

d) $y=\ln(1+x)$; h) $y=\ln(ax+b)$.

690. Using the Leibniz rule, find $y^{(n)}$, if:

a) $y=x\cdot e^x$; d) $y=\frac{1+x}{\sqrt{x}}$;

b) $y=x^2\cdot e^{-2x}$; e) $y=x^3 \ln x$.

c) $y=(1-x^2)\cos x$;

691. Find $f^{(n)}(0)$, if $f(x)=\ln\frac{1}{1-x}$

B. Higher-Order Derivatives of Functions Represented Parametrically and of Implicit Functions

In the following problems find $\frac{d^2y}{dx^2}$.

692. a) $\begin{cases} x=\ln t, \\ y=t^3; \end{cases}$ b) $\begin{cases} x=\arctan t, \\ y=\ln(1+t^2); \end{cases}$ c) $\begin{cases} x=\arcsin t \\ y=\sqrt{1-t^2}. \end{cases}$

693. a) $\begin{cases} x=a\cos t, \\ y=a\sin t; \end{cases}$ c) $\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t); \end{cases}$

b) $\begin{cases} x=a\cos^3 t, \\ y=a\sin^3 t; \end{cases}$ d) $\begin{cases} x=a(\sin t-t\cos t), \\ y=a(\cos t+t\sin t). \end{cases}$

694. a) $\begin{cases} x=\cos 2t, \\ y=\sin^2 t; \end{cases}$ **695.** a) $\begin{cases} x=\arctan t, \\ y=\frac{1}{2}t^2; \end{cases}$

b) $\begin{cases} x=e^{-at}, \\ y=e^{at}. \end{cases}$ b) $\begin{cases} x=\ln t, \\ y=\frac{1}{1-t}. \end{cases}$

696. Find $\frac{d^2x}{dy^2}$, if $\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$

697. Find $\frac{d^2y}{dx^2}$ for $t=0$, if $\begin{cases} x=\ln(1+t^2), \\ y=t^2. \end{cases}$

698. Show that y (as a function of x) defined by the equations $x=\sin t$, $y=ae^{t\sqrt{2}}+be^{-t\sqrt{2}}$ for any constants a and b satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2y.$$

In the following examples find $y'''=\frac{d^3y}{dx^3}$.

699. $\begin{cases} x=\sec t, \\ y=\tan t. \end{cases}$

700. $\begin{cases} x=e^{-t}\cos t, \\ y=e^{-t}\sin t. \end{cases}$

701. $\begin{cases} x=e^{-t}, \\ y=t^3. \end{cases}$

702. Find $\frac{d^ny}{dx^n}$, if $\begin{cases} x=\ln t, \\ y=t^m. \end{cases}$

703. Knowing the function $y=f(x)$, find the derivatives x'', x''' of the inverse function $x=f^{-1}(y)$.

704. Find y'', if $x^2+y^2=1$.

Solution. By the rule for differentiating a composite function we have $2x+2yy'=0$; whence $y'=-\frac{x}{y}$ and $y''=-\left(\frac{x}{y}\right)'_x=-\frac{y-xy'}{y^2}$. Substituting the value of y', we finally get:

$$y''=-\frac{y^2+x^2}{y^3}=-\frac{1}{y^3}.$$

In the following examples it is required to determine the derivative y'' of the function $y=f(x)$ represented implicitly.

705. $y^2=2px$.

706. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

707. $y=x+\arctan y$.

708. Having the equation $y=x+\ln y$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.

709. Find y'' at the point (1,1) if

$$x^2+5xy+y^2-2x+y-6=0.$$

710. Find y'' at (0,1) if

$$x^4-xy+y^4=1.$$

711. a) The function y is defined implicitly by the equation

$$x^2+2xy+y^2-4x+2y-2=0.$$

Find $\frac{d^3y}{dx^3}$ at the point (1,1).

b) Find $\frac{d^3y}{dx^3}$, if $x^2+y^2=a^2$.

Sec. 6. Differentials of First and Higher Orders

1°. **First-order differential.** The *differential (first-order) of a function* $y=f(x)$ is the principal part of its increment, which part is linear relative to the increment $\Delta x=dx$ of the independent variable x. The differential of a

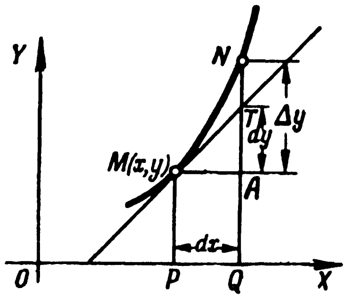

Fig. 19

function is equal to the product of its derivative by the differential of the independent variable

$$dy=y'dx,$$

whence

$$y'=\frac{dy}{dx}.$$

If MN is an arc of the graph of the function $y=f(x)$ (Fig. 19), MT is the tangent at $M(x, y)$ and

$$PQ=\Delta x=dx,$$

then the increment in the ordinate of the tangent

$$AT=dy$$

and the segment $AN=\Delta y$.

Example 1. Find the increment and the differential of the function $y=3x^2-x$.

Solution. First method:

$$\Delta y=3(x+\Delta x)^2-(x+\Delta x)-3x^2+x$$

or

$$\Delta y=(6x-1)\Delta x+3(\Delta x)^2.$$

Hence,

$$dy=(6x-1)\Delta x=(6x-1)dx.$$

Second method:

$$y'=6x-1;\ dy=y'dx=(6x-1)dx.$$

Example 2. Calculate Δy and dy of the function $y=3x^2-x$ for $x=1$ and $\Delta x=0.01$.

Solution. $\Delta y=(6x-1)\cdot\Delta x+3(\Delta x)^2=5\cdot 0.01+3\cdot(0.01)^2=0.0503$

and

$$dy=(6x-1)\Delta x=5\cdot 0.01=0.0500.$$

2°. Principal properties of differentials.
1) $dc=0$, where $c=\text{const}$.
2) $dx=\Delta x$, where x is an independent variable.
3) $d(cu)=c\,du$.
4) $d(u \pm v)=du \pm dv$.
5) $d(uv)=u\,dv+v\,du$.
6) $d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}$ $(v \neq 0)$.
7) $df(u)=f'(u)\,du$.

3°. Applying the differential to approximate calculations. If the increment Δx of the argument x is small in absolute value, then the differential dy of the function $y=f(x)$ and the increment Δy of the function are approximately equal:

$$\Delta y \approx dy,$$

that is,

$$f(x+\Delta x)-f(x) \approx f'(x)\,\Delta x,$$

whence

$$f(x+\Delta x) \approx f(x)+f'(x)\,dx.$$

Example 3. By how much (approximately) does the side of a square change if its area increases from 9 m^2 to 9.1 m^2?

Solution. If x is the area of the square and y is its side, then

$$y=\sqrt{x}.$$

It is given that $x=9$ and $\Delta x=0.1$.

The increment Δy in the side of the square may be calculated approximately as follows:

$$\Delta y \approx dy=y'\,\Delta x=\frac{1}{2\sqrt{9}}\cdot 0.1=0.016 \text{ m}.$$

4°. Higher-order differentials. A *second-order differential* is the differential of a first-order differential:

$$d^2y=d(dy).$$

We similarly define the *differentials of the third* and higher orders.

If $y=f(x)$ and x is an independent variable, then

$$\begin{aligned} d^2y &= y''(dx)^2, \\ d^3y &= y'''(dx)^3, \\ & \ldots\ldots\ldots \\ & \ldots\ldots\ldots \\ d^ny &= y^{(n)}(dx)^n. \end{aligned}$$

But if $y=f(u)$, where $u=\varphi(x)$, then

$$\begin{aligned} d^2y &= y''(du)^2+y'\,d^2u, \\ d^3y &= y'''(du)^3+3y''\,du\cdot d^2u+y'\,d^3u \end{aligned}$$

and so forth. (Here the primes denote derivatives with respect to u).

712. Find the increment Δy and the differential dy of the function $y=5x+x^2$ for $x=2$ and $\Delta x=0.001$.

713. Without calculating the derivative, find

$$d(1-x^3)$$

for $x=1$ and $\Delta x=-\frac{1}{3}$.

714. The area of a square S with side x is given by $S=x^2$. Find the increment and the differential of this function and explain the geometric significance of the latter.

715. Give a geometric interpretation of the increment and differential of the following functions:

a) the area of a circle, $S=\pi x^2$;

b) the volume of a cube, $v=x^3$.

716. Show that when $\Delta x \to 0$, the increment in the function $y=2^x$, corresponding to an increment Δx in x, is, for any x, equivalent to the expression $2^x \ln 2\, \Delta x$.

717. For what value of x is the differential of the function $y=x^2$ not equivalent to the increment in this function as $\Delta x \to 0$?

718. Has the function $y=|x|$ a differential for $x=0$?

719. Using the derivative, find the differential of the function $y=\cos x$ for $x=\frac{\pi}{6}$ and $\Delta x=\frac{\pi}{36}$.

720. Find the differential of the function

$$y=\frac{2}{\sqrt{x}}$$

for $x=9$ and $\Delta x=-0.01$.

721. Calculate the differential of the function

$$y=\tan x$$

for $x=\frac{\pi}{3}$ and $\Delta x=\frac{\pi}{180}$.

In the following problems find the differentials of the given functions for arbitrary values of the argument and its increment.

722. $y=\frac{1}{x^m}$.

723. $y=\frac{x}{1-x}$.

724. $y=\arcsin\frac{x}{a}$.

725. $y=\arctan\frac{x}{a}$.

726. $y=e^{-x^2}$.

727. $y=x\ln x-x$.

728. $y=\ln\frac{1-x}{1+x}$.

729. $r=\cot\varphi+\operatorname{cosec}\varphi$.

730. $s=\arctan e^t$.

731 Find dy if $x^2+2xy-y^2=a^2$.

Solution. Taking advantage of the invariancy of the form of a differential, we obtain $2x\,dx+2(y\,dx+x\,dy)-2y\,dy=0$

Whence

$$dy=-\frac{x+y}{x-y}\,dx.$$

In the following examples find the differentials of the functions defined implicitly.

732. $(x+y)^2 \cdot (2x+y)^3 = 1$.

733. $y = e^{-\frac{x}{y}}$.

734. $\ln \sqrt{x^2+y^2} = \arctan \frac{y}{x}$.

735. Find dy at the point (1,2), if $y^3 - y = 6x^2$.

736. Find the approximate value of sin 31°.

Solution. Putting $x = \operatorname{arc} 30° = \frac{\pi}{6}$ and $\Delta x = \operatorname{arc} 1° = \frac{\pi}{180}$, from formula (1) (see 3°) we have $\sin 31° \approx \sin 30° + \frac{\pi}{180} \cos 30° = 0.500 + 0.017 \cdot \frac{\sqrt{3}}{2} = 0.515$.

737. Replacing the increment of the function by the differential, calculate approximately:

a) $\cos 61°$; d) $\ln 0.9$;
b) $\tan 44°$; e) $\arctan 1.05$.
c) $e^{0.2}$;

738. What will be the approximate increase in the volume of a sphere if its radius $R = 15$ cm increases by 2 mm?

739. Derive the approximate formula (for $|\Delta x|$ that are small compared to x)

$$\sqrt{x+\Delta x} \approx \sqrt{x} + \frac{\Delta x}{2\sqrt{x}}.$$

Using it, approximate $\sqrt{5}$, $\sqrt{17}$, $\sqrt{70}$, $\sqrt{640}$.

740. Derive the approximate formula

$$\sqrt[3]{x+\Delta x} \approx \sqrt[3]{x} + \frac{\Delta x}{3\sqrt[3]{x^2}}$$

and find approximate values for $\sqrt[3]{10}$, $\sqrt[3]{70}$, $\sqrt[3]{200}$.

741. Approximate the functions:

a) $y = x^3 - 4x^2 + 5x + 3$ for $x = 1.03$;

b) $f(x) = \sqrt{1+x}$ for $x = 0.2$;

c) $f(x) = \sqrt[3]{\frac{1-x}{1+x}}$ for $x = 0.1$;

d) $y = e^{1-x^2}$ for $x = 1.05$.

742. Approximate tan 45°3′20″.

743. Find the approximate value of arc sin 0.54.

744. Approximate $\sqrt[4]{17}$.

745. Using Ohm's law, $I=\frac{E}{R}$, show that a small change in the current, due to a small change in the resistance, may be found approximately by the formula

$$\Delta I=-\frac{I}{R}\Delta R.$$

746. Show that, in determining the length of the radius, a relative error of 1% results in a relative error of approximately 2% in calculating the area of a circle and the surface of a sphere.

747. Compute d^2y, if $y=\cos 5x$.

Solution. $d^2y=y''(dx^2)=-25\cos 5x\,(dx)^2$.

748. $u=\sqrt{1-x^2}$, find d^2u.

749. $y=\arccos x$, find d^2y.

750. $y=\sin x\ln x$, find d^2y.

751. $z=\frac{\ln x}{x}$, find d^2z.

752. $z=x^2e^{-x}$, find d^3z.

753. $z=\frac{x^4}{2-x}$, find d^4z.

754. $u=3\sin(2x+5)$, find d^nu.

755. $y=e^{x\cos\alpha}\sin(x\sin\alpha)$, find d^ny.

Sec. 7. Mean-Value Theorems

1°. **Rolle's theorem.** If a function $f(x)$ is continuous on the interval $a\leqslant x\leqslant b$, has a derivative $f'(x)$ at every interior point of this interval, and

$$f(a)=f(b),$$

then the argument x has at least one value ξ, where $a<\xi<b$, such that

$$f'(\xi)=0.$$

2°. **Lagrange's theorem.** If a function $f(x)$ is continuous on the interval $a\leqslant x\leqslant b$ and has a derivative at every interior point of this interval, then

$$f(b)-f(a)=(b-a)f'(\xi),$$

where $a<\xi<b$.

3°. **Cauchy's theorem.** If the functions $f(x)$ and $F(x)$ are continuous on the interval $a\leqslant x\leqslant b$ and for $a<x<b$ have derivatives that do not vanish simultaneously, and $F(b)\neq F(a)$, then

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(\xi)}{F'(\xi)}, \quad \text{where } a<\xi<b.$$

756. Show that the function $f(x)=x-x^3$ on the intervals $-1\leqslant x\leqslant 0$ and $0\leqslant x\leqslant 1$ satisfies the Rolle theorem. Find the appropriate values of ξ.

Solution. The function $f(x)$ is continuous and differentiable for all values of x, and $f(-1)=f(0)=f(1)=0$. Hence, the Rolle theorem is applicable on the intervals $-1 \leqslant x \leqslant 0$ and $0 \leqslant x < 1$. To find ξ we form the equation $f'(x)=1-3x^2=0$. Whence $\xi_1=-\sqrt{\frac{1}{3}}$; $\xi_2=\sqrt{\frac{1}{3}}$, where $-1<\xi_1<0$ and $0<\xi_2<1$.

757. The function $f(x)=\sqrt[3]{(x-2)^2}$ takes on equal values $f(0)=f(4)=\sqrt[3]{4}$ at the end-points of the interval [0.4]. Does the Rolle theorem hold for this function on [0.4]?

758. Does the Rolle theorem hold for the function

$$f(x)=\tan x$$

on the interval $[0, \pi]$?

759. Let

$$f(x)=x(x+1)(x+2)(x+3).$$

Show that the equation

$$f'(x)=0$$

has three real roots.

760. The equation

$$e^x=1+x$$

obviously has a root $x=0$. Show that this equation cannot have any other real root.

761. Test whether the Lagrange theorem holds for the function

$$f(x)=x-x^3$$

on the interval $[-2,1]$ and find the appropriate intermediate value of ξ.

Solution. The function $f(x)=x-x^3$ is continuous and differentiable for all values of x, and $f'(x)=1-3x^2$ Whence, by the Lagrange formula, we have $f(1)-f(-2)=0-6=[1-(-2)]f'(\xi)$, that is, $f'(\xi)=-2$ Hence, $1-3\xi^2=-2$ and $\xi=\pm 1$; the only suitable value is $\xi=-1$, for which the inequality $-2<\xi<1$ holds

762. Test the validity of the Lagrange theorem and find the appropriate intermediate point ξ for the function $f(x)=x^{4/3}$ on the interval $[-1,1]$.

763. Given a segment of the parabola $y=x^2$ lying between two points $A(1,1)$ and $B(3,9)$, find a point the tangent to which is parallel to the chord AB.

764. Using the Lagrange theorem, prove the formula

$$\sin(x+h)-\sin x=h\cos\xi,$$

where $x<\xi<x+h$.

765. a) For the functions $f(x)=x^2+2$ and $F(x)=x^3-1$ test whether the Cauchy theorem holds on the interval [1,2] and find ξ;

b) do the same with respect to $f(x)=\sin x$ and $F(x)=\cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Sec. 8. Taylor's Formula

If a function $f(x)$ is continuous and has continuous derivatives up to the $(n-1)$th order inclusive on the interval $a \leqslant x \leqslant b$ (or $b \leqslant x \leqslant a$), and there is a finite derivative $f^{(n)}(x)$ at each interior point of the interval, then *Taylor's formula*

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$$
$$\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(\xi),$$

where $\xi=a+\theta(x-a)$ and $0<\theta<1$, holds true on the interval.

In particular, when $a=0$ we have (*Maclaurin's formula*)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^{(n)}(\xi),$$

where $\xi=\theta x$, $0<\theta<1$.

766. Expand the polynomial $f(x)=x^3-2x^2+3x+5$ in positive integral powers of the binomial $x-2$.

Solution. $f'(x)=3x^2-4x+3$; $f''(x)=6x-4$; $f'''(x)=6$; $f^{(n)}(x)=0$ for $n \geqslant 4$. Whence

$$f(2)=11;\ f'(2)=7;\ f''(2)=8;\ f'''(2)=6.$$

Therefore,

$$x^3-2x^2+3x+5=11+(x-2)\cdot 7+\frac{(x-2)^2}{2!}\cdot 8+\frac{(x-2)^3}{3!}\cdot 6$$

or

$$x^3-2x^2+3x+5=11+7(x-2)+4(x-2)^2+(x-2)^3.$$

767. Expand the function $f(x)=e^x$ in powers of $x+1$ to the term containing $(x+1)^3$.

Solution. $f^{(n)}(x)=e^x$ for all n, $f^{(n)}(-1)=\frac{1}{e}$. Hence,

$$e^x=\frac{1}{e}+(x+1)\frac{1}{e}+\frac{(x+1)^2}{2!}\frac{1}{e}+\frac{(x+1)}{3!}\frac{1}{e}+\frac{(x+1)^4}{4!}e^{\xi},$$

where $\xi=-1+\theta(x+1)$; $0<\theta<1$.

768. Expand the function $f(x)=\ln x$ in powers of $x-1$ up to the term with $(x-1)^2$.

769. Expand $f(x)=\sin x$ in powers of x up to the term containing x^3 and to the term containing x^5.

770. Expand $f(x)=e^x$ in powers of x up to the term containing x^{n-1}.

771. Show that $\sin(a+h)$ differs from

$$\sin a + h\cos a$$

by not more than $1/2\,h^2$.

772. Determine the origin of the approximate formulas:

a) $\sqrt{1+x}\approx 1+\frac{1}{2}x-\frac{1}{8}x^2, \quad |x|<1,$

b) $\sqrt[3]{1+x}\approx 1+\frac{1}{3}x-\frac{1}{9}x^2, \quad |x|<1$

and evaluate their errors.

773. Evaluate the error in the formula

$$e\approx 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}.$$

774. Due to its own weight, a heavy suspended thread lies in a catenary line $y=a\cosh\frac{x}{a}$. Show that for small $|x|$ the shape of the thread is approximately expressed by the parabola

$$y=a+\frac{x^2}{2a}.$$

775*. Show that for $|x|\ll a$, to within $\left(\frac{x}{a}\right)^2$, we have the approximate equality

$$e^{\frac{x}{a}}\approx\sqrt{\frac{a+x}{a-x}}.$$

Sec. 9. The L'Hospital-Bernoulli Rule for Evaluating Indeterminate Forms

1°. Evaluating the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Let the single-valued functions $f(x)$ and $\varphi(x)$ be differentiable for $0<|x-a|<h$; the derivative of one of them does not vanish.

If $f(x)$ and $\varphi(x)$ are both infinitesimals or both infinites as $x\to a$; that is, if the quotient $\frac{f(x)}{\varphi(x)}$, at $x=a$, is one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

provided that the limit of the ratio of derivatives exists.

The rule is also applicable when $a=\infty$.

If the quotient $\frac{f'(x)}{\varphi'(x)}$ again yields an indeterminate form, at the point $x=a$, of one of the two above-mentioned types and $f'(x)$ and $\varphi'(x)$ satisfy all the requirements that have been stated for $f(x)$ and $\varphi(x)$, we can then pass to the ratio of second derivatives, etc.

However, it should be borne in mind that the limit of the ratio $\frac{f(x)}{\varphi(x)}$ may exist, whereas the ratios of the derivatives do not tend to any limit (see Example 809).

2°. **Other indeterminate forms.** To evaluate an indeterminate form like $0\cdot\infty$, transform the appropriate product $f_1(x)\cdot f_2(x)$, where $\lim\limits_{x\to a} f_1(x)=0$ and $\lim\limits_{x\to a} f_2(x)=\infty$, into the quetient $\frac{f_1(x)}{\frac{1}{f_2(x)}}$ (the form $\frac{0}{0}$ (or $\frac{f_2(x)}{\frac{1}{f_1(x)}}$ (the form $\frac{\infty}{\infty}$).

In the case of the indeterminate form $\infty-\infty$, one should transform the appropriate difference $f_1(x)-f_2(x)$ into the product $f_1(x)\left[1-\frac{f_2(x)}{f_1(x)}\right]$ and first evaluate the indeterminate form $\frac{f_2(x)}{f_1(x)}$; if $\lim\limits_{x\to a}\frac{f_2(x)}{f_1(x)}=1$, then we reduce the expression to the form

$$\frac{1-\frac{f_2(x)}{f_1(x)}}{\frac{1}{f_1(x)}} \quad \text{(the form } \frac{0}{0}\text{)}.$$

The indeterminate forms 1^∞, 0^0, ∞^0 are evaluated by first taking logarithms and then finding the limit of the logarithm of the power $[f_1(x)]^{f_2(x)}$ (which requires evaluating a form like $0\cdot\infty$).

In certain cases it is useful to combine the L'Hospital rule with the finding of limits by elementary techniques.

Example 1. Compute

$$\lim_{x\to 0}\frac{\ln x}{\cot x} \quad \text{(form } \frac{\infty}{\infty}\text{)}.$$

Solution. Applying the L'Hospital rule we have

$$\lim_{x\to 0}\frac{\ln x}{\cot x}=\lim_{x\to 0}\frac{(\ln x)'}{(\cot x)'}=-\lim_{x\to 0}\frac{\sin^2 x}{x}.$$

We get the indeterminate form $\frac{0}{0}$; however, we do not need to use the L'Hospital rule, since

$$\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x=1\cdot 0=0.$$

We thus finally get

$$\lim_{x\to 0}\frac{\ln x}{\cot x}=0.$$

Example 2. Compute

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right) \text{ (form } \infty-\infty).$$

Reducing to a common denominator, we get

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x} \text{ (form } \frac{0}{0}).$$

Before applying the L'Hospital rule, we replace the denominator of the latter fraction by an equivalent infinitesimal (Ch. 1, Sec. 4) $x^2\sin^2 x\sim x^4$. We obtain

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4} \text{ (form } \frac{0}{0}).$$

The L'Hospital rule gives

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{2x-\sin 2x}{4x^3}=\lim_{x\to 0}\frac{2-2\cos 2x}{12x^2}.$$

Then, in elementary fashion, we find

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{1-\cos 2x}{6x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{6x^2}=\frac{1}{3}.$$

Example 3. Compute

$$\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}} \text{ (form } 1^\infty)$$

Taking logarithms and applying the L'Hospital rule, we get

$$\lim_{x\to 0}\ln(\cos 2x)^{\frac{3}{x^2}}=\lim_{x\to 0}\frac{3\ln\cos 2x}{x^2}=-6\lim_{x\to 0}\frac{\tan 2x}{2x}=-6.$$

Hence, $\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}}=e^{-6}$.

Find the indicated limits of functions in the following examples.

776. $\lim_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}$.

Solution. $\lim_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}=\lim_{x\to 1}\frac{3x^2-4x-1}{3x^2-7}=\frac{1}{2}$.

777. $\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}$.

778. $\lim_{x\to 1}\frac{1-x}{1-\sin\frac{\pi x}{2}}$.

779. $\lim_{x\to 0}\frac{\cosh x-1}{1-\cos x}$.

780. $\lim_{x\to 0}\frac{\tan x-\sin x}{x-\sin x}$.

781. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$.

782. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\tan x}{\tan 5x}$.

783. $\lim\limits_{x\to\infty} \frac{e^x}{x^5}$.

784. $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

785. $\lim\limits_{x\to 0} \frac{\frac{\pi}{x}}{\cot\frac{\pi x}{2}}$.

786. $\lim\limits_{x\to 0} \frac{\ln(\sin mx)}{\ln\sin x}$.

787. $\lim\limits_{x\to 0} (1-\cos x)\cot x$.

Solution. $\lim\limits_{x\to 0} (1-\cos x)\cot x = \lim\limits_{x\to 0} \frac{(1-\cos x)\cos x}{\sin x} = \lim\limits_{x\to 0} \frac{(1-\cos x)\cdot 1}{\sin x} =$

$= \lim\limits_{x\to 0} \frac{\sin x}{\cos x} = 0$

788. $\lim\limits_{x\to 1} (1-x)\tan\frac{\pi x}{2}$.

789. $\lim\limits_{x\to 0} \operatorname{arc\,sin} x \cot x$.

790. $\lim\limits_{x\to 0} (x^n e^{-x})$, $n>0$.

791. $\lim\limits_{x\to\infty} x\sin\frac{a}{x}$.

792. $\lim\limits_{x\to\infty} x^n \sin\frac{a}{x}$, $n>0$.

793. $\lim\limits_{x\to 1} \ln x \ln(x-1)$.

794. $\lim\limits_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$.

Solution. $\lim\limits_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim\limits_{x\to 1} \frac{x\ln x - x + 1}{(x-1)\ln x} =$

$$= \lim_{x\to 1} \frac{x\cdot\frac{1}{x} + \ln x - 1}{\ln x + \frac{1}{x}(x-1)} = \lim_{x\to 1} \frac{\ln x}{\ln x - \frac{1}{x} + 1} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}.$$

795. $\lim\limits_{x\to 3} \left(\frac{1}{x-3} - \frac{5}{x^2-x-6}\right)$.

796. $\lim\limits_{x\to 1} \left[\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})}\right]$.

797. $\lim\limits_{x\to\frac{\pi}{2}} \left(\frac{x}{\cot x} - \frac{\pi}{2\cos x}\right)$.

798. $\lim\limits_{x\to 0} x^x$.

Solution. We have $x^x = y$; $\ln y = x\ln x$: $\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} x\ln x =$

$= \lim\limits_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$, whence $\lim\limits_{x\to 0} y = 1$, that is, $\lim\limits_{x\to 0} x^x = 1$.

799. $\lim\limits_{x\to+\infty} x^{\frac{1}{x}}$.

800. $\lim\limits_{x\to 0} x^{\frac{3}{4+\ln x}}$.

801. $\lim\limits_{x\to 0} x^{\sin x}$.

802. $\lim\limits_{x\to 1} (1-x)^{\cos\frac{\pi x}{2}}$.

803. $\lim\limits_{x\to 0} (1+x^2)^{\frac{1}{x}}$.

804. $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$.

805. $\lim\limits_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{4}}$.

806. $\lim\limits_{x\to 0} (\cot x)^{\frac{1}{\ln x}}$.

807. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$.

808. $\lim\limits_{x\to 0} (\cot x)^{\sin x}$.

809. Prove that the limits of

a) $\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=0$;

b) $\lim\limits_{x\to\infty}\frac{x-\sin x}{x+\sin x}=1$

cannot be found by the L'Hospital-Bernoulli rule. Find these limits directly.

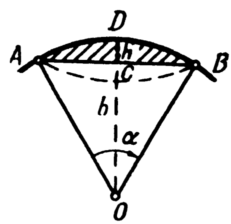

Fig. 20

810*. Show that the area of a circular segment with minor central angle α, which has a chord $AB=b$ and $CD=h$ (Fig. 20), is approximately

$$S\approx\frac{2}{3}bh$$

with an arbitrarily small relative error when $\alpha\to 0$.