Chapter III

THE EXTREMA OF A FUNCTION AND THE GEOMETRIC APPLICATIONS OF A DERIVATIVE

Sec. 1. The Extrema of a Function of One Argument

1°. **Increase and decrease of functions.** The function $y=f(x)$ is called *increasing (decreasing)* on some interval if, for any points x_1 and x_2 which belong to this interval, from the inequality $x_1<x_2$ we get the inequality $f(x_1)<f(x_2)$ (Fig. 21*a*) [$f(x_1)>f(x_2)$ (Fig. 21*b*)]. If $f(x)$ is continuous on the interval $[a, b]$ and $f'(x)>0$ [$f'(x)<0$] for $a<x<b$, then $f(x)$ increases (decreases) on the interval $[a, b]$.

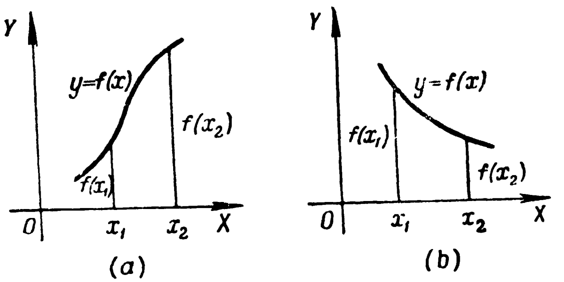

Fig. 21

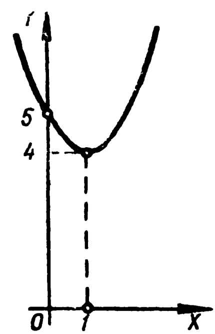

Fig. 22

In the simplest cases, the domain of definition of $f(x)$ may be subdivided into a finite number of intervals of increase and decrease of the function (*intervals of monotonicity*). These intervals are bounded by critical points x [where $f'(x)=0$ or $f'(x)$ does not exist].

Example 1. Test the following function for increase and decrease:

$$y=x^2-2x+5.$$

Solution. We find the derivative

$$y'=2x-2=2(x-1).$$

Whence $y'=0$ for $x=1$. On a number scale we get two intervals of monotonicity: $(-\infty, 1)$ and $(1, +\infty)$. From (1) we have: 1) if $-\infty<x<1$, then $y'<0$, and, hence, the function $f(x)$ decreases in the interval $(-\infty, 1)$; 2) if $1<x<+\infty$, then $y'>0$, and, hence, the function $f(x)$ increases in the interval $(1, +\infty)$ (Fig. 22).

Example 2. Determine the intervals of increase and decrease of the function

$$y=\frac{1}{x+2}.$$

Solution. Here, $x=-2$ is a discontinuity of the function and $y'=-\frac{1}{(x+2)^2}<0$ for $x\neq-2$. Hence, the function y decreases in the intervals $-\infty<x<-2$ and $-2<x<+\infty$.

Example 3. Test the following function for increase or decrease:

$$y=\frac{1}{5}x^5-\frac{1}{3}x^3.$$

Solution Here,

$$y'=x^4-x^2. \tag{2}$$

Solving the equation $x^4-x^2=0$, we find the points $x_1=-1$, $x_2=0$, $x_3=1$ at which the derivative y' vanishes. Since y' can change sign only when passing through points at which it vanishes or becomes discontinuous (in the given case, y' has no discontinuities), the derivative in each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0,1)$ and $(1, +\infty)$ retains its sign; for this reason, the function under investigation is monotonic in each of these intervals. To determine in which of the indicated intervals the function increases and in which it decreases, one has to determine the sign of the derivative in each of the intervals. To determine what the sign of y' is in the interval $(-\infty, -1)$, it is sufficient to determine the sign of y' at some point of the interval; for example, taking $x=-2$, we get from (2) $y'=12>0$, hence, $y'>0$ in the interval $(-\infty, -1)$ and the function in this interval increases Similarly, we find that $y'<0$ in the interval $(-1, 0)$ (as a check, we can take $x=-\frac{1}{2}$), $y'<0$ in the interval $(0,1)$ (here, we can use $x=1/2$) and $y'>0$ in the interval $(1, +\infty)$.

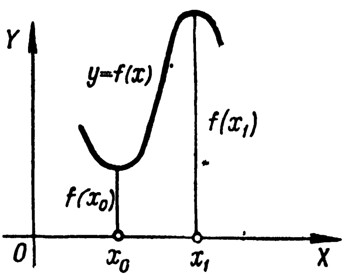

Fig 23

Thus, the function being tested increases in the interval $(-\infty, -1)$, decreases in the interval $(-1, 1)$ and again increases in the interval $(1, +\infty)$.

2°. **Extremum of a function.** If there exists a two-sided neighbourhood of a point x_0 such that for every point $x\neq x_0$ of this neighbourhood we have the inequality $f(x)>f(x_0)$, then the point x_0 is called the *minimum point* of the function $y=f(x)$, while the number $f(x_0)$ is called the *minimum* of the function $y=f(x)$. Similarly, if for any point $x\neq x_1$ of some neighbourhood of the point x_1, the inequality $f(x)<f(x_1)$ is fulfilled, then x_1 is called the *maximum point* of the function $f(x)$, and $f(x_1)$ is the *maximum* of the function (Fig. 23). The minimum point or maximum point of a function is its *extremal point* (bending point), and the minimum or maximum of a function is called the *extremum* of the function. If x_0 is an extremal point of the function $f(x)$, then $f'(x_0)=0$, or $f'(x_0)$ does not exist (necessary condition for the existence of an extremum). The converse is not true: points at which $f'(x)=0$, or $f'(x)$, does not exist (*critical points*) are not necessarily extremal points of the function $f(x)$.

The sufficient conditions for the existence and absence of an extremum of a continuous function $f(x)$ are given by the following rules:

1. If there exists a neighbourhood $(x_0-\delta, x_0+\delta)$ of a critical point x_0 such that $f'(x)>0$ for $x_0-\delta<x<x_0$ and $f'(x)<0$ for $x_0<x<x_0+\delta$, then x_0 is the maximum point of the function $f(x)$; and if $f'(x)<0$ for $x_0-\delta<x<x_0$ and $f'(x)>0$ for $x_0<x<x_0+\delta$, then x_0 is the minimum point of the function $f(x)$.

Finally, if there is some positive number δ such that $f'(x)$ retains its sign unchanged for $0<|x-x_0|<\delta$, then x_0 is not an extremal point of the function $f(x)$.

2. If $f'(x_0)=0$ and $f''(x_0)<0$, then x_0 is the maximum point; if $f'(x_0)=0$ and $f''(x_0)>0$, then x_0 is the minimum point; but if $f'(x_0)=0$, $f''(x_0)=0$, and $f'''(x_0)\neq 0$, then the point x_0 is not an extremal point.

More generally: let the first of the derivatives (not equal to zero at the point x_0) of the function $f(x)$ be of the order k. Then, if k is even, the point x_0 is an extremal point, namely, the maximum point, if $f^{(k)}(x_0)<0$; and it is the minimum point, if $f^{(k)}(x_0)>0$ But if k is odd, then x_0 is not an extremal point.

Example 4. Find the extrema of the function

$$y=2x+3\sqrt[3]{x^2}.$$

Solution. Find the derivative

$$y'=2+\frac{2}{\sqrt[3]{x}}=\frac{2}{\sqrt[3]{x}}(\sqrt[3]{x}+1). \tag{3}$$

Equating the derivative y' to zero, we get:

$$\sqrt[3]{x}+1=0.$$

Whence, we find the critical point $x_1=-1$. From formula (3) we have: if $x=-1-h$, where h is a sufficiently small positive number, then $y'>0$; but if $x=-1+h$, then $y'<0$*). Hence, $x_1=-1$ is the maximum point of the function y, and $y_{\max}=1$.

Equating the denominator of the expression of y' in (3) to zero, we get

$$\sqrt[3]{x}=0;$$

whence we find the second critical point of the function $x_2=0$, where there is no derivative y'. For $x=-h$, we obviously have $y'<0$; for $x=h$ we have $y'>0$. Consequently, $x_2=0$ is the minimum point of the function y, and $y_{\min}=0$ (Fig. 24). It is also possible to test the behaviour of the function at the point $x=-1$ by means of the second derivative

$$y''=-\frac{2}{3x\sqrt[3]{x}}.$$

Here, $y''<0$ for $x_1=-1$ and, hence, $x_1=-1$ is the maximum point of the function.

3°. **Greatest and least values.** The least (greatest) value of a continuous function $f(x)$ on a given interval $[a, b]$ is attained either at the critical points of the function or at the end-points of the interval $[a, b]$.

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*) If it is difficult to determine the sign of the derivative y', one can calculate arithmetically by taking for h a sufficiently small positive number.

Example 5. Find the greatest and least values of the function

$$y = x^3 - 3x + 3$$

on the interval $-1^1/_2 \leqslant x \leqslant 2^1/_2$.

Solution. Since

$$y' = 3x^2 - 3,$$

it follows that the critical points of the function y are $x_1 = -1$ and $x_2 = 1$.

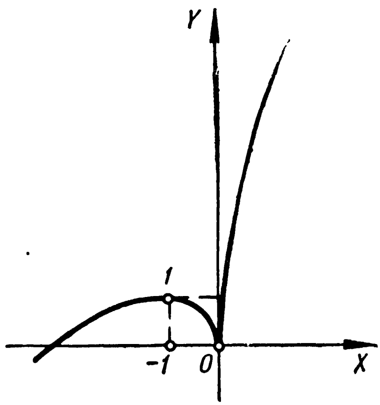

Fig. 24

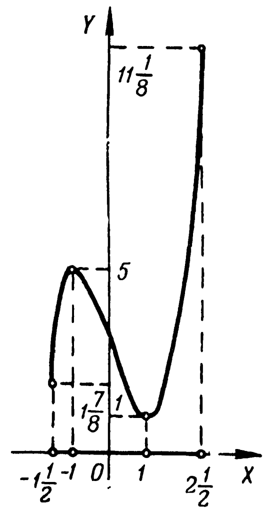

Fig. 25

Comparing the values of the function at these points and the values of the function at the end-points of the given interval

$$y(-1) = 5; \; y(1) = 1; \; y\left(-1\frac{1}{2}\right) = 4\frac{1}{8}; \quad y\left(2\frac{1}{2}\right) = 11\frac{1}{8},$$

we conclude (Fig. 25) that the function attains its least value, $m = 1$, at the point $x = 1$ (at the minimum point), and the greatest value $M = 11\frac{1}{8}$ at the point $x = 2^1/_2$ (at the right-hand end-point of the interval).

Determine the intervals of decrease and increase of the functions:

811. $y = 1 - 4x - x^2$.

812. $y = (x - 2)^2$.

813. $y = (x + 4)^3$.

814. $y = x^2(x - 3)$.

815. $y = \frac{x}{x-2}$.

816. $y = \frac{1}{(x-1)^2}$.

817. $y = \frac{x}{x^2 - 6x - 16}$.

818. $y = (x - 3)\sqrt{x}$.

819. $y=\frac{x}{3}-\sqrt[3]{x}$.

820. $y=x+\sin x$.

821. $y=x\ln x$.

822. $y=\arcsin(1+x)$.

823. $y=2e^{x^2-4x}$.

824. $y=2^{\frac{1}{x-a}}$.

825. $y=\frac{e^x}{x}$.

Test the following functions for extrema:

826. $y=x^2+4x+6$.

Solution. We find the derivative of the given function, $y'=2x+4$. Equating y' to zero, we get the critical value of the argument $x=-2$. Since $y'<0$ when $x<-2$, and $y'>0$ when $x>-2$, it follows that $x=-2$ is the minimum point of the function, and $y_{\min}=2$. We get the same result by utilizing the sign of the second derivative at the critical point $y''=2>0$.

827. $y=2+x-x^2$.

828. $y=x^3-3x^2+3x+2$.

829. $y=2x^3+3x^2-12x+5$.

Solution. We find the derivative

$$y'=6x^2+6x-12=6(x^2+x-2).$$

Equating the derivative y' to zero, we get the critical points $x_1=-2$ and $x_2=1$. To determine the nature of the extremum, we calculate the second derivative $y''=6(2x+1)$. Since $y''(-2)<0$, it follows that $x_1=-2$ is the maximum point of the function y, and $y_{\max}=25$. Similarly, we have $y''(1)>0$; therefore, $x_2=1$ is the minimum point of the function y and $y_{\min}=-2$.

830. $y=x^2(x-12)^2$.

831. $y=x(x-1)^2(x-2)^3$.

832. $y=\frac{x^3}{x^2+3}$.

833. $y=\frac{x^2-2x+2}{x-1}$.

834. $y=\frac{(x-2)(8-x)}{x^2}$.

835. $y=\frac{16}{x(4-x^2)}$.

836. $y=\frac{4}{\sqrt{x^2+8}}$.

837. $y=\frac{x}{\sqrt[3]{x^2-4}}$.

838. $y=\sqrt[3]{(x^2-1)^2}$.

839. $y=2\sin 2x+\sin 4x$.

840. $y=2\cos\frac{x}{2}+3\cos\frac{x}{3}$.

841. $y=x-\ln(1+x)$.

842. $y=x\ln x$.

843. $y=x\ln^2 x$.

844. $y=\cosh x$.

845. $y=xe^x$.

846. $y=x^2e^{-x}$.

847. $y=\frac{e^x}{x}$.

848. $y=x-\arctan x$.

Determine the least and greatest values of the functions on the indicated intervals (if the interval is not given, determine the greatest and least values of the function throughout the domain of definition).

849. $y=\frac{x}{1+x^2}$.

850. $y=\sqrt{x(10-x)}$.

851. $y=\sin^4 x+\cos^4 x$.

852. $y=\arccos x$.

853. $y=x^3$ on the interval $[-1,3]$.

854. $y=2x^3+3x^2-12x+1$
a) on the interval $[-1,5]$;
b) on the interval $[-10,12]$.

855. Show that for positive values of x we have the inequality

$$x+\frac{1}{x}\geqslant 2.$$

856. Determine the coefficients p and q of the quadratic trinomial $y=x^2+px+q$ so that this trinomial should have a minimum $y=3$ when $x=1$. Explain the result in geometrical terms.

857. Prove the inequality

$$e^x > 1+x \quad \text{when } x\neq 0.$$

Solution. Consider the function

$$f(x)=e^x-(1+x).$$

In the usual way we find that this function has a single minimum $f(0)=0$. Hence,

$$f(x)>f(0) \quad \text{when } x\neq 0,$$
$$\text{and so } e^x>1+x \quad \text{when } x\neq 0,$$

as we set out to prove.

Prove the inequalities:

858. $x-\frac{x^3}{6}<\sin x<x$ when $x>0$.

859. $\cos x>1-\frac{x^2}{2}$ when $x\neq 0$.

860. $x-\frac{x^2}{2}<\ln(1+x)<x$ when $x>0$.

861. Separate a given positive number a into two summands such that their product is the greatest possible.

862. Bend a piece of wire of length l into a rectangle so that the area of the latter is greatest.

863. What right triangle of given perimeter $2p$ has the greatest area?

864. It is required to build a rectangular playground so that it should have a wire net on three sides and a long stone wall on the fourth. What is the optimum (in the sense of area) shape of the playground if l metres of wire netting are available?

865. It is required to make an open rectangular box of greatest capacity out of a square sheet of cardboard with side a by cutting squares at each of the angles and bending up the ends of the resulting cross-like figure.

866. An open tank with a square base must have a capacity of v litres. What size will it be if the least amount of tin is used?

867. Which cylinder of a given volume has the least overall surface?

868. In a given sphere inscribe a cylinder with the greatest volume.

869. In a given sphere inscribe a cylinder having the greatest lateral surface.

870. In a given sphere inscribe a cone with the greatest volume.

871. Inscribe in a given sphere a right circular cone with the greatest lateral surface.

872. About a given cylinder circumscribe a right cone of least volume (the planes and centres of their circular bases coincide).

873. Which of the cones circumscribed about a given sphere has the least volume?

874. A sheet of tin of width a has to be bent into an open cylindrical channel (Fig. 26). What should the central angle φ be so that the channel will have maximum capacity?

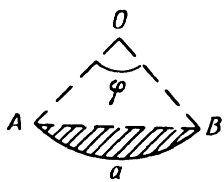

Fig. 26

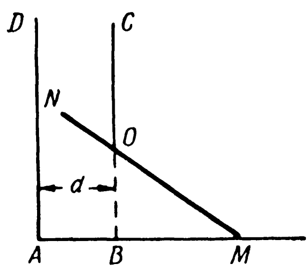

Fig. 27

875. Out of a circular sheet cut a sector such that when made into a funnel it will have the greatest possible capacity.

876. An open vessel consists of a cylinder with a hemisphere at the bottom; the walls are of constant thickness. What will the dimensions of the vessel be if a minimum of material is used for a given capacity?

877. Determine the least height $h=OB$ of the door of a vertical tower $ABCD$ so that this door can pass a rigid rod MN of length l, the end of which, M, slides along a horizontal straight line AB. The width of the tower is $d<l$ (Fig. 27).

878. A point $M_0(x_0, y_0)$ lies in the first quadrant of a coordinate plane. Draw a straight line through this point so that the triangle which it forms with the positive semi-axes is of least area.

879. Inscribe in a given ellipse a rectangle of largest area with sides parallel to the axes of the ellipse.

880. Inscribe a rectangle of maximum area in a segment of the parabola $y^2 = 2px$ cut off by the straight line $x = 2a$.

881. On the curve $y = \frac{1}{1+x^2}$ find a point at which the tangent forms with the x-axis the greatest (in absolute value) angle.

882. A messenger leaving A on one side of a river has to get to B on the other side. Knowing that the velocity along the bank is k times that on the water, determine the angle at which the messenger has to cross the river so as to reach B in the shortest possible time. The width of the river is h and the distance between A and B along the bank is d.

883. On a straight line $AB = a$ connecting two sources of light A (of intensity p) and B (of intensity q), find the point M that receives least light (the intensity of illumination is inversely proportional to the square of the distance from the light source).

884. A lamp is suspended above the centre of a round table of radius r. At what distance should the lamp be above the table so that an object on the edge of the table will get the greatest illumination? (The intensity of illumination is directly proportional to the cosine of the angle of incidence of the light rays and is inversely proportional to the square of the distance from the light source.)

885. It is required to cut a beam of rectangular cross-section out of a round log of diameter d. What should the width x and the height y be of this cross-section so that the beam will offer maximum resistance a) to compression and b) to bending?

Note. The resistance of a beam to compression is proportional to the area of its cross-section, to bending—to the product of the width of the cross-section by the square of its height.

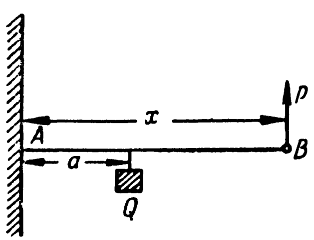

Fig. 2

886. A homogeneous rod AB, which can rotate about a point A (Fig. 28), is carrying a load Q kilograms at a distance of a cm from A and is held in equilibrium by a vertical force P applied to the free end B of the rod. A linear centimetre of the rod weighs q kilograms. Determine the length of the rod x so that the force P should be least, and find $P_{\min}$.

887*. The centres of three elastic spheres A, B, C are situated on a single straight line. Sphere A of mass M moving with velocity v strikes B, which, having acquired a certain velocity, strikes C of mass m. What mass should B have so that C will have the greatest possible velocity?

888. N identical electric cells can be formed into a battery in different ways by combining n cells in series and then combining the resulting groups $\left(\text{the number of groups is } \frac{N}{n}\right)$ in parallel. The current supplied by this battery is given by the formula

$$I=\frac{Nn\mathcal{E}}{NR+n^2r},$$

where $\mathcal{E}$ is the electromotive force of one cell, r is its internal resistance, and R is its external resistance.

For what value of n will the battery produce the greatest current?

889. Determine the diameter y of a circular opening in the body of a dam for which the discharge of water per second Q will be greatest, if $Q=cy\sqrt{h-y}$, where h is the depth of the lowest point of the opening (h and the empirical coefficient c are constant).

890. If $x_1, x_2, \ldots, x_n$ are the results of measurements of equal precision of a quantity x, then its most probable value will be that for which the sum of the squares of the errors

$$\sigma=\sum_{i=1}^{n}(x-x_i)^2$$

is of least value (the principle of least squares).

Prove that the most probable value of x is the arithmetic mean of the measurements.

Sec. 2. The Direction of Concavity. Points of Inflection

1°. The concavity of the graph of a function. We say that the graph of a differentiable function $y=f(x)$ is *concave down* in the interval (a,b) [*concave up* in the interval (a_1,b_1)] if for $a<x<b$ the arc of the curve is below (or for $a_1<x<b_1$, above) the tangent drawn at any point of the interval (a,b) or of the interval (a_1,b_1)] (Fig. 29). A sufficient condition for the concavity downwards (upwards) of a graph $y=f(x)$ is that the following inequality be fulfilled in the appropriate interval:

$$f''(x)<0\ [f''(x)>0].$$

2°. Points of inflection. A point $[x_0, f(x_0)]$ at which the direction of concavity of the graph of some function changes is called a *point of inflection* (Fig. 29).

For the abscissa of the point of inflection x_0 of the graph of a function $y=f(x)$ there is no second derivative $f''(x_0)=0$ or $f''(x_0)$. Points at which $f''(x)=0$ or $f''(x)$ does not exist are called *critical points of the second kind*. The critical point of the second kind x_0 is the abscissa of the point of inflection if $f''(x)$ retains constant signs in the intervals $x_0-\delta<x<x_0$ and $x_0<x<x_0+\delta$, where δ is some positive number; provided these signs are opposite. And it is not a point of inflection if the signs of $f''(x)$ are the same in the above-indicated intervals.

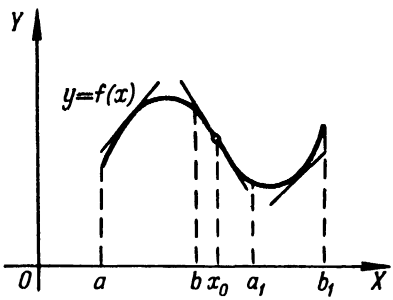

Fig. 29

Example 1. Determine the intervals of concavity and convexity and also the points of inflection of the Gaussian curve

$$y=e^{-x^2}.$$

Solution. We have

$$y'=-2xe^{-x}$$

and

$$y''=(4x^2-2)\,e^{-x^2}.$$

Equating the second derivative y'' to zero, we find the critical points of the second kind

$$x_1=-\frac{1}{\sqrt{2}} \quad \text{and} \quad x_2=\frac{1}{\sqrt{2}}.$$

These points divide the number scale $-\infty<x<+\infty$ into three intervals: I $(-\infty,\ x_1)$, II $(x_1,\ x_2)$, and III $(x_2,\ +\infty)$. The signs of y'' will be, respectively, $+$, $-$, $+$ (this is obvious if, for example, we take one point in each of the intervals and substitute the corresponding values of x into y'') Therefore: 1) the curve is concave up when $-\infty<x<-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}<x<+\infty$; 2) the curve is concave down when $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$. The points $\left(\frac{\pm 1}{\sqrt{2}},\ \frac{1}{\sqrt{e}}\right)$ are points of inflection (Fig. 30).

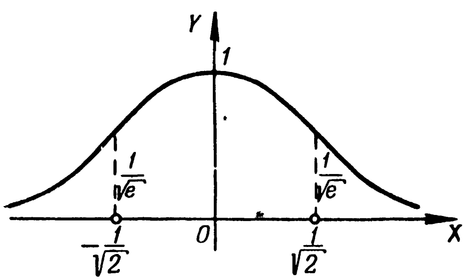

Fig. 30

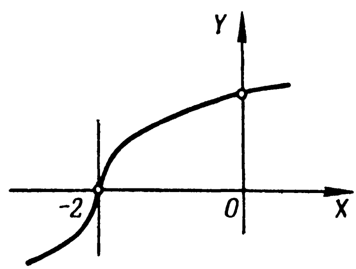

Fig. 31

It will be noted that due to the symmetry of the Gaussian curve about the y-axis, it would be sufficient to investigate the sign of the concavity of this curve on the semiaxis $0<x<+\infty$ alone.

Example 2. Find the points of inflection of the graph of the function

$$y=\sqrt[3]{x+2}.$$

Solution. We have:

$$y''=-\frac{2}{9}(x+2)^{-\frac{5}{3}}=\frac{-2}{9\sqrt[3]{(x+2)^5}}. \qquad (1)$$

It is obvious that y'' does not vanish anywhere.

Equating to zero the denominator of the fraction on the right of (1), we find that y'' does not exist for $x=-2$. Since $y''>0$ for $x<-2$ and $y''<0$ for $x>-2$, it follows that $(-2,0)$ is the point of inflection (Fig. 31). The tangent at this point is parallel to the axis of ordinates, since the first derivative y' is infinite at $x=-2$.

Find the intervals of concavity and the points of inflection of the graphs of the following functions:

891. $y=x^3-6x^2+12x+4$.

892. $y=(x+1)^4$.

893. $y=\frac{1}{x+3}$.

894. $y=\frac{x^3}{x^2+12}$.

895. $y=\sqrt[3]{4x^3-12x}$.

896. $y=\cos x$.

897. $y=x-\sin x$.

898. $y=x^2\ln x$.

899. $y=\arctan x-x$.

900. $y=(1+x^2)e^x$.

Sec. 3. Asymptotes

1°. **Definition.** If a point (x,y) is in continuous motion along a curve $y=f(x)$ in such a way that at least one of its coordinates approaches infinity (and at the same time the distance of the point from some straight line tends to zero), then this straight line is called an *asymptote* of the curve.

2°. **Vertical asymptotes.** If there is a number a such that

$$\lim_{x\to a} f(x)=\pm\infty,$$

then the straight line $x=a$ is an asymptote *(vertical asymptote)*.

3° **Inclined asymptotes.** If there are limits

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k_1$$

and

$$\lim_{x\to+\infty}[f(x)-k_1x]=b_1,$$

then the straight line $y=k_1x+b_1$ will be an asymptote (a *right inclined asymptote* or, when $k_1=0$, a *right horizontal asymptote*).

If there are limits

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k_2$$

and

$$\lim_{x \to -\infty} [f(x) - k_2 x] = b_2,$$

then the straight line $y = k_2 x + b_2$ is an asymptote (a *left inclined asymptote* or, when $k_2 = 0$, a *left horizontal asymptote*). The graph of the function $y = f(x)$ (we assume the function is single-valued) cannot have more than one right (inclined or horizontal) and more than one left (inclined or horizontal) asymptote.

Example 1. Find the asymptotes of the curve

$$y = \frac{x^2}{\sqrt{x^2 - 1}}.$$

Solution. Equating the denominator to zero, we get two vertical asymptotes:

$$x = -1 \quad \text{and} \quad x = 1.$$

We seek the inclined asymptotes. For $x \to +\infty$ we obtain

$$k_1 = \lim_{x \to +\infty} \frac{y}{x} = \lim_{x \to +\infty} \frac{x^2}{x\sqrt{x^2 - 1}} = 1,$$

$$b_1 = \lim_{x \to +\infty} (y - x) = \lim_{x \to +\infty} \frac{x^2 - x\sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} = 0,$$

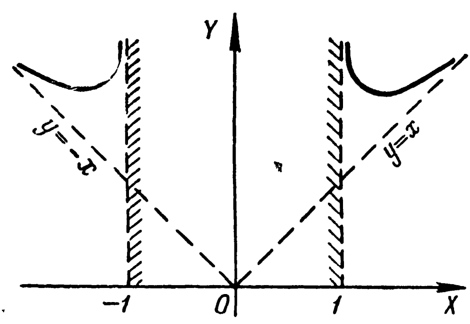

Fig. 32

hence, the straight line $y = x$ is the right asymptote. Similarly, when $x \to -\infty$, we have

$$k_2 = \lim_{x \to -\infty} \frac{y}{x} = -1;$$

$$b_2 = \lim_{x \to -\infty} (y + x) = 0.$$

Thus, the left asymptote is $y = -x$ (Fig. 32). Testing a curve for asymptotes is simplified if we take into consideration the symmetry of the curve.

Example 2. Find the asymptotes of the curve

$$y = x + \ln x.$$

Solution. Since

$$\lim_{x\to+0} y=-\infty,$$

the straight line $x=0$ is a vertical asymptote (lower). Let us now test the curve only for the inclined right asymptote (since $x>0$).

We have:

$$k=\lim_{x\to+\infty}\frac{y}{x}=1,$$

$$b=\lim_{x\to+\infty}(y-x)=\lim_{x\to+\infty}\ln x=\infty.$$

Hence, there is no inclined asymptote.

If a curve is represented by the parametric equations $x=\varphi(t)$, $y=\psi(t)$, then we first test to find out whether there are any values of the parameter t for which one of the functions $\varphi(t)$ or $\psi(t)$ becomes infinite, while the other remains finite. When $\varphi(t_0)=\infty$ and $\psi(t_0)=c$, the curve has a horizontal asymptote $y=c$. When $\psi(t_0)=\infty$ and $\varphi(t_0)=c$, the curve has a vertical asymptote $x=c$.

If $\varphi(t_0)=\psi(t_0)=\infty$ and

$$\lim_{t\to t_0}\frac{\psi(t)}{\varphi(t)}=k;\quad \lim_{t\to t_0}[\psi(t)-k\varphi(t)]=b,$$

then the curve has an inclined asymptote $y=kx+b$.

If the curve is represented by a polar equation $r=f(\varphi)$, then we can find its asymptotes by the preceding rule after transforming the equation of the curve to the parametric form by the formulas $x=r\cos\varphi=f(\varphi)\cos\varphi$; $y=r\sin\varphi=f(\varphi)\sin\varphi$.

Find the asymptotes of the following curves:

901. $y=\frac{1}{(x-2)^2}$.

902. $y=\frac{x}{x^2-4x+3}$.

903. $y=\frac{x^2}{x^2-4}$.

904. $y=\frac{x^3}{x^2+9}$.

905. $y=\sqrt{x^2-1}$.

906. $y=\frac{x}{\sqrt{x^2+3}}$.

907. $y=\frac{x^2+1}{\sqrt{x^2-1}}$.

908. $y=x-2+\frac{x^2}{\sqrt{x^2+9}}$.

909. $y=e^{-x^2}+2$.

910. $y=\frac{1}{1-e^x}$.

911. $y=e^{\frac{1}{x}}$.

912. $y=\frac{\sin x}{x}$.

913. $y=\ln(1+x)$.

914. $x=t$; $y=t+2\arctan t$.

915. Find the asymptote of the hyperbolic spiral $r=\frac{a}{\varphi}$.

Sec. 4. Graphing Functions by Characteristic Points

In constructing the graph of a function, first find its domain of definition and then determine the behaviour of the function on the boundary of this domain. It is also useful to note any peculiarities of the function (if there are any), such as symmetry, periodicity, constancy of sign, monotonicity, etc.

Then find any points of discontinuity, bending points, points of inflection, asymptotes, etc. These elements help to determine the general nature of the graph of the function and to obtain a mathematically correct outline of it.

Example 1. Construct the graph of the function

$$y=\frac{x}{\sqrt[3]{x^2-1}}.$$

Solution. a) The function exists everywhere except at the points $x=\pm 1$. The function is odd, and therefore the graph is symmetric about the point $O(0, 0)$. This simplifies construction of the graph

b) The discontinuities are $x=-1$ and $x=1$; and $\lim\limits_{x\to 1\pm 0} y=\pm\infty$ and $\lim\limits_{x\to -1\pm 0} y=\pm\infty$; hence, the straight lines $x=\pm 1$ are vertical asymptotes of the graph.

c) We seek inclined asymptotes, and find

$$k_1=\lim_{x\to+\infty}\frac{y}{x}=0,$$

$$b_1=\lim_{x\to+\infty} y=\infty,$$

thus, there is no right asymptote. From the symmetry of the curve it follows that there is no left-hand asymptote either.

d) We find the critical points of the first and second kinds, that is, points at which the first (or, respectively, the second) derivative of the given function vanishes or does not exist.

We have:

$$y'=\frac{x^2-3}{3\sqrt[3]{(x^2-1)^4}}, \tag{1}$$

$$y''=\frac{2x(9-x^2)}{9\sqrt[3]{(x^2-1)^7}}. \tag{2}$$

The derivatives y' and y'' are nonexistent only at $x=\pm 1$, that is, only at points where the function y itself does not exist; and so the critical points are only those at which y' and y'' vanish.

From (1) and (2) it follows that

$$y'=0 \quad \text{when } x=\pm\sqrt{3};$$
$$y''=0 \quad \text{when } x=0 \text{ and } x=\pm 3.$$

Thus, y' retains a constant sign in each of the intervals $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, -1)$, $(-1, 1)$, $(1, \sqrt{3})$ and $(\sqrt{3}, +\infty)$, and y''—in each of the intervals $(-\infty, -3)$, $(-3, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 3)$ and $(3, +\infty)$.

To determine the signs of y' (or, respectively, y'') in each of the indicated intervals, it is sufficient to determine the sign of y' (or y'') at some one point of each of these intervals.

It is convenient to tabulate the results of such an investigation (Table I), calculating also the ordinates of the characteristic points of the graph of the function. It will be noted that due to the oddness of the function y, it is enough to calculate only for $x \geqslant 0$; the left-hand half of the graph is constructed by the principle of odd symmetry.

Table I

x	0	(0, 1)	1	$(1, \sqrt{3})$	$\sqrt{3} \approx 1.73$	$(\sqrt{3}, 3)$	3	$(3, +\infty)$
y	0	$-$	$\pm\infty$	$+$	$\frac{\sqrt{3}}{\sqrt[3]{2}} \approx 1.37$	$+$	1.5	$+$
y'	$-$	$-$	non-exist	$-$	0	$+$	$+$	$+$
y''	0	$-$	non-exist	$+$	$+$	$+$	0	$-$
Conclusions	Point of inflection	Function decreases, graph is concave down	Discontinuit,	Function decreases, graph is concave up	Min. point	Function increases; graph is concave up	Point of inflection	Function increases; graph is concave down

e) Using the results of the investigation, we construct the graph of the function (Fig 33).

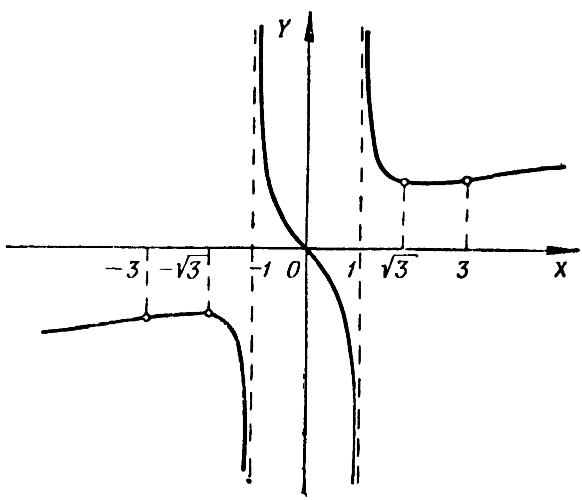

Fig. 33

Example 2. Graph the function

$$y=\frac{\ln x}{x}.$$

Solution. a) The domain of definition of the function is $0<x<+\infty$.

b) There are no discontinuities in the domain of definition, but as we approach the boundary point ($x=0$) of the domain of definition we have

$$\lim_{x\to 0} y=\lim_{x\to 0}\frac{\ln x}{x}=-\infty.$$

Hence, the straight line $x=0$ (ordinate axis) is a vertical asymptote.

c) We seek the right asymptote (there is no left asymptote, since x cannot tend to $-\infty$):

$$k=\lim_{x\to+\infty}\frac{y}{x}=0;$$

$$b=\lim_{x\to+\infty} y=0.$$

The right asymptote is the axis of abscissas: $y=0$.

d) We find the critical points; and have

$$y'=\frac{1-\ln x}{x^2},$$

$$y''=\frac{2\ln x-3}{x^3};$$

y' and y'' exist at all points of the domain of definition of the function and

$$y'=0 \quad\text{when } \ln x=1,\ \text{that is, when } x=e;$$

$$y''=0 \quad\text{when } \ln x=\frac{3}{2},\ \text{that is, when } x=e^{3/2}.$$

We form a table, including the characteristic points (Table II). In addition to the characteristic points it is useful to find the points of intersection of

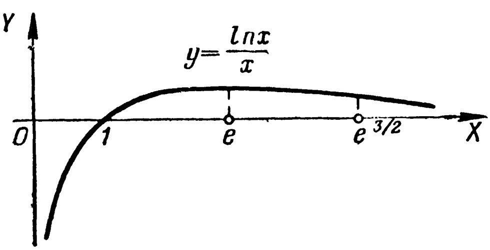

Fig. 34

the curve with the coordinate axes. Putting $y=0$, we find $x=1$ (the point of intersection of the curve with the axis of abscissas); the curve does not intersect the axis of ordinates

e) Utilizing the results of investigation, we construct the graph of the function (Fig. 34).

Table II

x	0	(0, 1)	1	(1, e)	$e \approx 2.72$	$\left(e, e^{\frac{3}{2}}\right)$	$e^{\frac{3}{2}} \approx 4.49$	$\left(e^{\frac{3}{2}}, +\infty\right)$
y	$-\infty$	$-$	0	$+$	$\frac{1}{e} \approx 0.37$	$+$	$\frac{3}{2\sqrt{e^3}} \approx 0.33$	$+$
y'	nonexist.	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	nonexist	$-$	$-$	$-$	$-$	$-$	0	$+$
Conclusions	Boundary point of domain of def. of function Vertical asymptote	Funct increases graph is concave down	Funct incr. graph is concave down	Funct incr., graph is concave down	Max point of funct.	Funct decr., graph is concave down	Point of inflection	Function decreases; graph is concave up

Graph the following functions and determine for each function its domain of definition, discontinuities, extremal points, intervals of increase and decrease, points of inflection of its graph, the direction of concavity, and also the asymptotes.

916. $y=x^3-3x^2$.

917. $y=\frac{6x^2-x^4}{9}$.

918. $y=(x-1)^2(x+2)$.

919. $y=\frac{(x-2)^2(x+4)}{4}$.

920. $y=\frac{(x^2-5)^3}{125}$.

921. $y=\frac{x^2-2x+2}{x-1}$.

922. $y=\frac{x^4-3}{x}$.

923. $y=\frac{x^4+3}{x}$.

924. $y=x^2+\frac{2}{x}$.

925. $y=\frac{1}{x^2+3}$.

926. $y=\frac{8}{x^2-4}$.

927. $y=\frac{4x}{4+x^2}$.

928. $y=\frac{4x-12}{(x-2)^2}$.

929. $y=\frac{x}{x^2-4}$.

930. $y=\frac{16}{x^2(x-4)}$.

931. $y=\frac{3x^4+1}{x^3}$.

932. $y=\sqrt{x}+\sqrt{4-x}$.

933. $y=\sqrt{8+x}-\sqrt{8-x}$.

934. $y=x\sqrt{x+3}$.

935. $y=\sqrt{x^3-3x}$.

936. $y=\sqrt[3]{1-x^2}$.

937. $y=\sqrt[3]{1-x^3}$.

938. $y=2x+2-3\sqrt[3]{(x+1)^2}$.

939. $y=\sqrt[3]{x+1}-\sqrt[3]{x-1}$.

940. $y=\sqrt[3]{(x+4)^2}-\sqrt[3]{(x-4)^2}$.

941. $y=\sqrt[3]{(x-2)^2}+\sqrt[3]{(x-4)^2}$.

942. $y=\frac{4}{\sqrt{4-x^2}}$.

943. $y=\frac{8}{x\sqrt{x^2-4}}$.

944. $y=\frac{x}{\sqrt[3]{x^2-1}}$.

945. $y=\frac{x}{\sqrt[3]{(x-2)^2}}$.

946. $y=xe^{-x}$.

947. $y=\left(a+\frac{x^2}{a}\right)e^{\frac{x}{a}}$.

948. $y=e^{8x-x^2-14}$.

949. $y=(2+x^2)e^{-x^2}$.

950. $y=2|x|-x^2$.

951. $y=\frac{\ln x}{\sqrt{x}}$.

952. $y=\frac{x^2}{2}\ln\frac{x}{a}$.

953. $y=\frac{x}{\ln x}$.

954. $y=(x+1)\ln^2(x+1)$.

955. $y=\ln(x^2-1)+\frac{1}{x^2-1}$.

956. $y=\ln\frac{\sqrt{x^2+1}-1}{x}$.

957. $y=\ln(1+e^{-x})$.

958. $y=\ln\left(e+\frac{1}{x}\right)$.

959. $y=\sin x+\cos x$.

960. $y=\sin x+\frac{\sin 2x}{2}$.

961. $y=\cos x-\cos^2 x$.

962. $y=\sin^3 x+\cos^3 x$.

963. $y=\frac{1}{\sin x+\cos x}$.

964. $y=\frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}$.

965. $y=\sin x\cdot\sin 2x$.

966. $y=\cos x\cdot\cos 2x$.

967. $y=x+\sin x$.

968. $y=\arcsin(1-\sqrt[3]{x^2})$.

969. $y=\frac{\arcsin x}{\sqrt{1-x^2}}$.

970. $y=2x-\tan x$.

971. $y=x\arctan x$.

972. $y=x\arctan\frac{1}{x}$ when $x\neq 0$ and $y=0$ when $x=0$.

973. $y=x-2\,\mathrm{arc\,cot}\,x$.

974. $y=\frac{x}{2}+\arctan x$.

975. $y=\ln\sin x$.

976. $y=\mathrm{arc\,cosh}\left(x+\frac{1}{x}\right)$.

977. $y=e^{\sin x}$.

978. $y=e^{\arcsin\sqrt{x}}$.

979. $y=e^{\arctan x}$.

980. $y=\ln\sin x$.

981. $y=\ln\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

982. $y=\ln x-\arctan x$.

983. $y=\cos x-\ln\cos x$.

984. $y=\arctan(\ln x)$.

985. $y=\arcsin\ln(x^2+1)$.

986. $y=x^x$.

987. $y=x^{\frac{1}{x}}$.

A good exercise is to graph the functions indicated in Examples 826-848.

Construct the graphs of the following functions represented parametrically.

988. $x=t^2-2t,\quad y=t^2+2t$.

989. $x=a\cos^3 t,\quad y=a\sin t\quad (a>0)$.

990. $x=te^t,\quad y=te^{-t}$.

991. $x=t+e^{-t},\quad y=2t+e^{-2t}$.

992. $x=a(\sinh t-t),\quad y=a(\cosh t-1)\quad (a>0)$.

Sec. 5. Differential of an Arc. Curvature

1°. Differential of an arc. The differential of an arc s of a plane curve represented by an equation in Cartesian coordinates x and y is expressed by the formula

$$ds=\sqrt{(dx)^2+(dy)^2};$$

here, if the equation of the curve is of the form

a) $y=f(x)$, then $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$;

b) $x=f_1(y)$, then $ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$;

c) $x=\varphi(t)$, $y=\psi(t)$, then $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$;

d) $F(x, y)=0$, then $ds=\frac{\sqrt{F_x'^2+F_y'^2}}{|F_y'|}\,dx=\frac{\sqrt{F_x'^2+F_y'^2}}{|F_x'|}\,dy$.

Denoting by α the angle formed by the tangent (in the direction of increasing arc of the curve s) with the positive x-direction, we get

$$\cos\alpha=\frac{dx}{ds},$$

$$\sin\alpha=\frac{dy}{ds}.$$

In polar coordinates,

$$ds=\sqrt{(dr)^2+(r\,d\varphi)^2}=\sqrt{r^2+\left(\frac{dr}{d\varphi}\right)^2}\,d\varphi$$

Denoting by β the angle between the radius vector of the point of the curve and the tangent to the curve at this point, we have

$$\cos\beta=\frac{dr}{ds},$$

$$\sin\beta=r\frac{d\varphi}{ds}.$$

2°. Curvature of a curve. The *curvature* K of a curve at one of its points M is the limit of the ratio of the angle between the positive directions of the tangents at the points M and N of the curve *(angle of contingence)* to the length of the arc $\overset{\smile}{MN}=\Delta s$ when $N\to M$ (Fig. 35), that is,

$$K=\lim_{\Delta s\to 0}\frac{\Delta\alpha}{\Delta s}=\frac{d\alpha}{ds},$$

where α is the angle between the positive directions of the tangent at the point M and the x-axis.

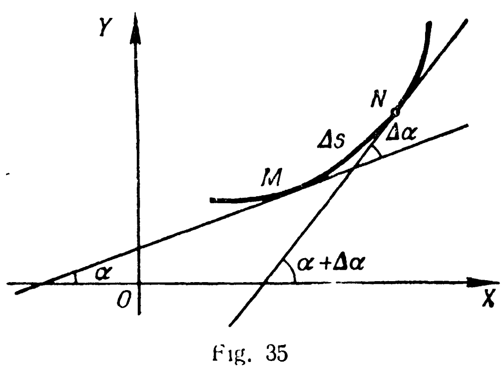

Fig. 35

The *radius of curvature* R is the reciprocal of the absolute value of the curvature, i. e.,

$$R=\frac{1}{|K|}.$$

The circle $\left(K=\frac{1}{a}\right.$, where a is the radius of the circle) and the straight line ($K=0$) are lines of constant curvature.

We have the following formulas for computing the curvature in rectangular coordinates (accurate to within the sign):

1) if the curve is given by an equation explicitly, $y=f(x)$, then

$$K=\frac{y''}{(1+y'^2)^{3/2}};$$

2) if the curve is given by an equation implicitly, $F(x, y)=0$, then

$$K=\frac{\begin{vmatrix} F''_{xx} & F''_{xy} & F'_x \\ F''_{yx} & F''_{yy} & F'_y \\ F'_x & F'_y & 0 \end{vmatrix}}{\left(F'^2_x+F'^2_y\right)^{3/2}};$$

3) if the curve is represented by equations in parametric form, $x=\varphi(t)$, $y=\psi(t)$, then

$$K=\frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2+y'^2)^{3/2}},$$

where

$$x'=\frac{dx}{dt}, \quad y'=\frac{dy}{dt}, \quad x''=\frac{d^2x}{dt^2}, \quad y''=\frac{d^2y}{dt^2}.$$

In polar coordinates, when the curve is given by the equation $r=f(\varphi)$, we have

$$K=\frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{3/2}},$$

where

$$r'=\frac{dr}{d\varphi} \quad \text{and} \quad r''=\frac{d^2r}{d\varphi^2}.$$

3°. **Circle of curvature.** The *circle of curvature* (or *osculating circle*) of a curve at the point M is the limiting position of a circle drawn through M and two other points of the curve, P and Q, as $P \to M$ and $Q \to M$.

The radius of the circle of curvature is equal to the radius of curvature, and the centre of the circle of curvature (the *centre of curvature*) lies on the normal to the curve drawn at the point M in the direction of concavity of the curve.

The coordinates X and Y of the centre of curvature of the curve are computed from the formulas

$$X=x-\frac{y'(1+y'^2)}{y''}, \quad Y=y+\frac{1+y'^2}{y''}.$$

The *evolute* of a curve is the locus of the centres of curvature of the curve.

If in the formulas for determining the coordinates of the centre of curvature we regard X and Y as the current coordinates of a point of the evolute, then these formulas yield parametric equations of the evolute with parameter x or y (or t, if the curve itself is represented by equations in parametric form)

Example 1. Find the equation of the evolute of the parabola $y=x^2$.

Solution. $X=-4x^3$, $Y=\frac{1+6x^2}{2}$. Eliminating the parameter x, we find the equation of the evolute in explicit form, $Y=\frac{1}{2}+3\left(\frac{X}{4}\right)^{2/3}$.

The *involute* of a curve is a curve for which the given curve is an evolute.

The normal MC of the involute Γ_2 is a tangent to the evolute Γ_1; the length of the arc $\overset{\frown}{CC_1}$ of the evolute is equal to the corresponding increment in the radius of curvature $\overset{\frown}{CC_1}=M_1C_1-MC$; that is why the involute Γ_2 is also called the *evolvent* of the curve Γ_1 obtained by unwinding a taut thread wound onto Γ_1 (Fig. 36). To each evolute there corresponds an infinitude of involutes, which are related to different initial lengths of thread.

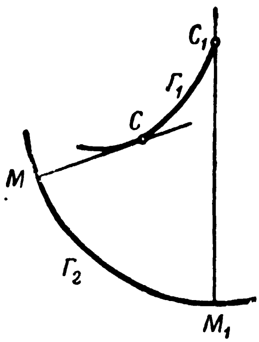

Fig. 36

4°. **Vertices of a curve.** The *vertex* of a curve is a point of the curve at which the curvature has a maximum or a minimum. To determine the vertices of a curve, we form the expression of the curvature K and find its extremal points. In place of the curvature K we can take the radius of curvature $R=\frac{1}{|K|}$ and seek its extremal points if the computations are simpler in this case.

Example 2. Find the vertex of the catenary $y=a\cosh\frac{x}{a}$ $(a>0)$.

Solution. Since $y'=\sinh\frac{x}{a}$ and $y''=\frac{1}{a}\cosh\frac{x}{a}$, it follows that $K=$ $=\frac{1}{a\cosh^2\frac{x}{a}}$ and, hence, $R=a\cosh^2\frac{x}{a}$. We have $\frac{dR}{dx}=\sinh 2\frac{x}{a}$. Equating the derivative $\frac{dR}{dx}$ to zero, we get $\sinh 2\frac{x}{a}=0$, whence we find the sole critical point $x=0$. Computing the second derivative $\frac{d^2R}{dx^2}$ and putting into it the value $x=0$, we get $\left.\frac{d^2R}{dx^2}\right|_{x=0}=\left.\frac{2}{a}\cosh 2\frac{x}{a}\right|_{x=0}=\frac{2}{a}>0$. Therefore, $x=0$ is the minimum point of the radius of curvature (or of the maximum of curvature) of the catenary. The vertex of the catenary $y=a\cosh\frac{x}{a}$ is, thus, the point $A\ (0,\ a)$.

Find the differential of the arc, and also the cosine and sine of the angle formed, with the positive x-direction, by the tangent to each of the following curves:

993. $x^2+y^2=a^2$ (circle).

994. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).

995 $y^2=2px$ (parabola).

996. $x^{2/3}+y^{2/3}=a^{2/3}$ (astroid).

997. $y=a\cosh\frac{x}{a}$ (catenary).

998. $x=a(t-\sin t)$; $y=a(1-\cos t)$ (cycloid).

999. $x=a\cos^3 t$, $y=a\sin^3 t$ (astroid).

Find the differential of the arc, and also the cosine or sine of the angle formed by the radius vector and the tangent to each of the following curves:

1000. $r=a\varphi$ (spiral of Archimedes).

1001. $r=\frac{a}{\varphi}$ (hyperbolic spiral).

1002. $r=a\sec^2\frac{\varphi}{2}$ (parabola).

1003. $r=a\cos^2\frac{\varphi}{2}$ (cardioid).

1004. $r=a^{\varphi}$ (logarithmic spiral).

1005. $r^2=a^2\cos 2\varphi$ (lemniscate).

Compute the curvature of the given curves at the indicated points:

1006. $y=x^4-4x^3-18x^2$ at the coordinate origin.

1007. $x^2+xy+y^2=3$ at the point (1, 1).

1008. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the vertices $A(a, 0)$ and $B(0, b)$.

1009. $x=t^2$, $y=t^3$ at the point (1, 1).

1010. $r^2=2a^2\cos 2\varphi$ at the vertices $\varphi=0$ and $\varphi=\pi$.

1011. At what point of the parabola $y^2=8x$ is the curvature equal to 0.128?

1012. Find the vertex of the curve $y=e^x$.

Find the radii of curvature (at any point) of the given lines:

1013. $y=x^3$ (cubic parabola).

1014. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).

1015. $x=\frac{y^2}{4}-\frac{\ln y}{2}$.

1016. $x=a\cos^3 t$; $y=a\sin^3 t$ (astroid).

1017. $x=a(\cos t+t\sin t)$; $y=a(\sin t-t\cos t)$ involute of a circle).

1018. $r=ae^{k\varphi}$ (logarithmic spiral).

1019. $r=a(1+\cos\varphi)$ (cardioid).

1020. Find the least value of the radius of curvature of the parabola $y^2=2px$.

1021. Prove that the radius of curvature of the catenary $y=a\cosh\frac{x}{a}$ is equal to a segment of the normal.

Compute the coordinates of the centre of curvature of the given curves at the indicated points:

1022. $xy=1$ at the point (1, 1).
1023. $ay^2=x^3$ at the point (a, a).

Write the equations of the circles of curvature of the given curves at the indicated points:

1024. $y=x^2-6x+10$ at the point (3, 1).
1025. $y=e^x$ at the point (0, 1).

Find the evolutes of the curves:

1026. $y^2=2px$ (parabola).
1027. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).
1028. Prove that the evolute of the cycloid

$$x=a(t-\sin t), \quad y=a(1-\cos t)$$

is a displaced cycloid.

1029. Prove that the evolute of the logarithmic spiral

$$r=ae^{k\varphi}$$

is also a logarithmic spiral with the same pole.

1030. Show that the curve (the *involute of a circle*)

$$x=a(\cos t+t\sin t), \quad y=a(\sin t-t\cos t)$$

is the involute of the circle $x=a\cos t$; $y=a\sin t$.